10. The Sinusoidal Steady-state Analysis

Input : Sinusoidal function

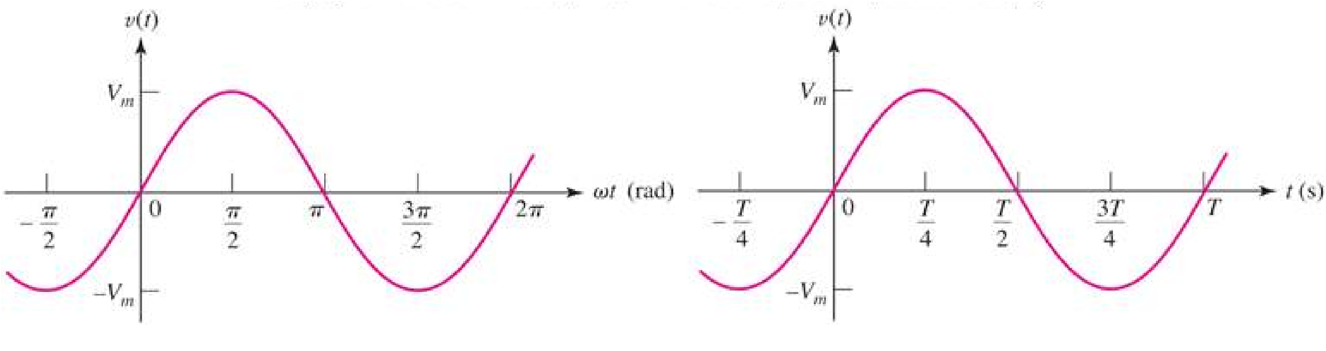
Natural response : same

Forced response (R, L and C): Resistance → Impedance(Z) in the s-domain

Extension of techniques of resistive circuit analysis.

10.1 Characteristics of Sinusoids

$v(t) = V_m \sin(wt)$



(a) (b)

(unit) Radian or Degree v.s second

$v(t) = V_m \sin(wt + \theta)$

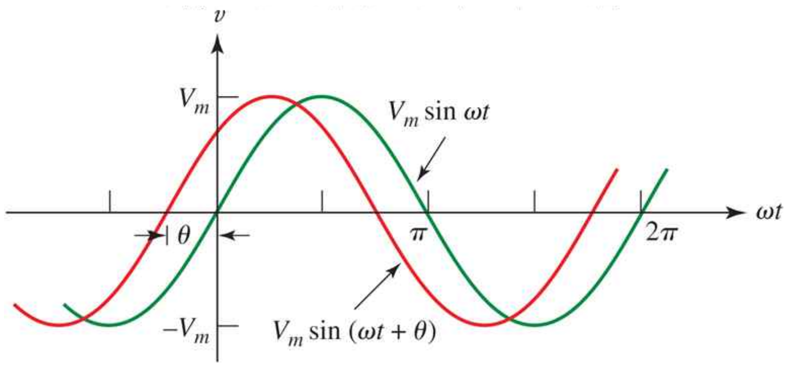
(1) Amplitude : V_m

(2) Radian (Angular) Frequency : w

$$w = 2\pi f = \frac{2\pi}{T} \quad \leftarrow f = \frac{1}{T}$$

rad/sec(각속도)

(3) Phase : θ



Example) Plot the waveforms v.s wt-axis and t-axis

(1) $v(t) = 10\sin(100\pi t)$

(2) $v(t) = 10\sin(100\pi t + 45^\circ)$

(2) $v(t) = 10\sin(100\pi t - 45^\circ)$

◆ Lagging and Leading

(1) $v(t) = 10\sin(100\pi t)$ and $v(t) = 10\sin(100\pi t + 45^\circ)$

(2) $v(t) = 100\sin(2{,}000\pi t)$ and $v(t) = 100\sin(2{,}000\pi t - \frac{\pi}{6})$

sol)

Out of phase : Lagging or Leading
In phase : The same phase

◆ Converting Sins to Cosines

$\cos(wt) = \sin(wt + 90^\circ)$
$\sin(wt \pm 180^\circ) = -\sin(wt)$ and $\cos(wt \pm 180^\circ) = -\cos(wt)$

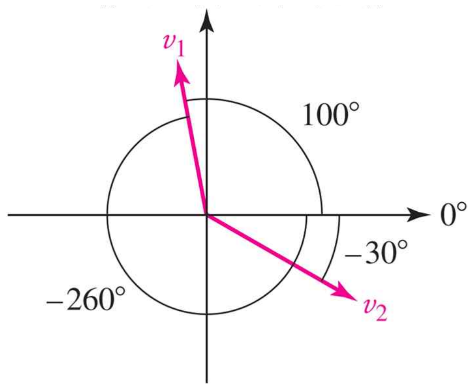
Example) Find phase difference.

$v_1(t) = V_{m1}\cos(5t + 10^\circ)$ and $v_2(t) = V_{m2}\sin(5t - 30^\circ)$

sol)

$$\begin{aligned} v_1(t) &= V_{m1}\cos(5t + 10^\circ) \\ &= V_{m1}\sin(5t + 10^\circ + 90^\circ) \\ &= V_{m1}\sin(5t + 100^\circ) \\ &= V_{m1}\sin(5t - 30^\circ + 130^\circ) \end{aligned}$$

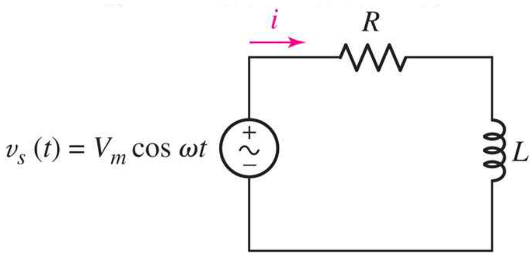
Hence, $v_1(t)$ leads $v_2(t)$ by 130°.
OR $v_1(t)$ lags $v_2(t)$ by 230°.



10.2 Forced Response to Sinusoidal Functions

Natural response : the same (Source-free)

◆ The State-state response



$$L\frac{di}{dt} + Ri = V_m \cos(wt)$$

sol)

The forced response always have the general form

$$i(t) = I_1 \cos(wt) + I_2 \sin(wt)$$
$$= \text{Original form} + \text{its Derivatives}$$

$$L[-I_1 w \sin(wt) + I_2 w \cos(wt)] + R[I_1 \cos(wt) + I_2 \sin(wt)] = V_m \cos(wt)$$

$$[-LI_1 w + RI_2] \sin(wt) + [LI_2 w + RI_1 - V_m] \cos(wt) = 0$$

$$\Rightarrow \ -LI_1 w + RI_2 = 0 \quad \text{AND} \quad LI_2 w + RI_1 - V_m = 0$$

$$\Rightarrow \ I_1 = \frac{RV_m}{R^2 + w^2 L^2} \ \text{ and } \ I_2 = \frac{wLV_m}{R^2 + w^2 L^2}$$

Hence, $i(t) = \dfrac{RV_m}{R^2 + w^2 L^2} cos(wt) + \dfrac{wLV_m}{R^2 + w^2 L^2} sin(wt)$

$$= \frac{V_m}{\sqrt{R^2 + w^2 L^2}} cos(wt - \theta)$$

where $\theta = \tan^{-1}(\frac{wL}{R})$, $A\cos(wt) + B\sin(wt) = \dfrac{1}{\sqrt{A^2 + B^2}} cos(wt - \theta)$

Example 10.1)

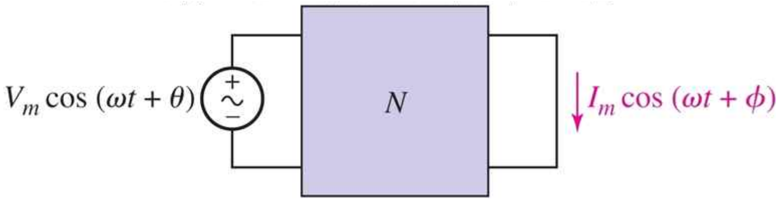
skip

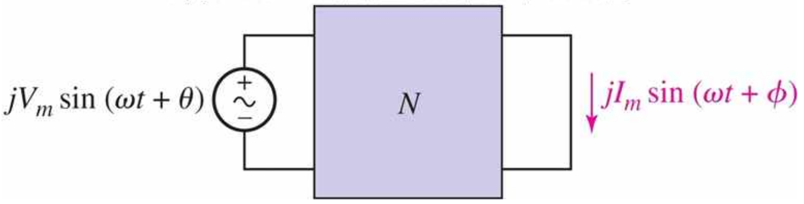
using the other methods

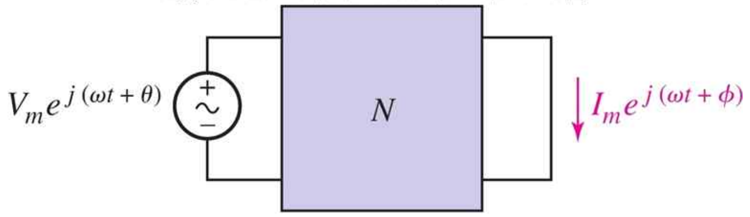
10.3 complex forcing fn.

$e^{i\theta} = \cos(\theta) + j\sin(\theta)$

$V_m e^{j(wt+\theta)} = V_m\cos(w+\theta) + jV_m\sin(w+\theta)$







$V_m\cos(w+\theta) = Re\{V_m e^{j(wt+\theta)}\}$ and $I_m\cos(w+\theta) = Re\{I_m e^{j(wt+\theta)}\}$

The use of complex quantities in the sinusoidal steady-state analysis leads to methods which are much simpler than that of real quantities.

Example) The previous example

sol)

If the complex forcing function is applied

$v_s(t) \rightarrow V_m e^{jwt}$

then, complex response with unknown amplitude I_m and phase angle ϕ.

$i(t) \rightarrow I_m e^{j(wt+\phi)}$ $\therefore)$ $i(t) = I_m\cos(wt+\phi)$

Find amplitude I_m and phase ϕ.

$$Ri+L\frac{di}{dt}=v_s(t) \qquad \rightarrow \quad R\left[I_m e^{j(wt+\phi)}\right]+L\frac{d}{dt}\left[I_m e^{j(wt+\phi)}\right]=V_m e^{jwt}$$

$$RI_m e^{jwt} e^{j\phi}+LI_m e^{j\phi}\left[jw\, e^{jwt}\right]=V_m e^{jwt}$$

$$RI_m e^{j\phi}+jwLI_m e^{j\phi}=V_m$$

$$I_m e^{j\phi}=\frac{V_m}{R+jwL}=\frac{V_m e^{j0^\circ}}{\sqrt{R^2+w^2L^2}\ e^{j\tan^{-1}(wL/R)}}$$

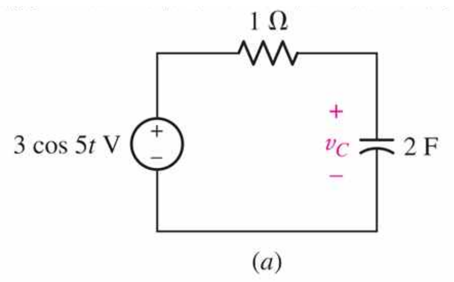
$$=\frac{V_m}{\sqrt{R^2+w^2L^2}}\ e^{-j\tan^{-1}(wL/R)}$$

Hence, $I_m=\dfrac{V_m}{\sqrt{R^2+w^2L^2}}$ and $\phi=-\tan^{-1}\left(\dfrac{wL}{R}\right)$

$$\Rightarrow\ i(t)=I_m\cos(wt+\phi)$$

$$=\frac{V_m}{\sqrt{R^2+w^2L^2}}cos(wt-\tan^{-1}[wL/R])$$

Example 10.2) Find steady-state $v_c(t)$



sol)

$v_s(t)=3\cos(5t)\ \rightarrow\ 3e^{j5t}$

$v_c(t)=V_m\cos(5t+\phi)\ \rightarrow\ V_m e^{j(5t+\phi)}$

Differential eq.

$$Ri_c(t)+v_c(t)=v_s(t)\ \leftarrow\ i_c=C\frac{dv_c}{dt}$$

$$RC\frac{dv_c}{dt}+v_c(t)=v_s(t)$$

$$2\frac{dv_c}{dt}+v_c=v_s$$

$v_s(t)\rightarrow 3e^{j5t}$ and $v_c(t)\rightarrow V_m e^{j(5t+\phi)}$

$$2\left[V_m e^{j\phi}(j5)e^{j5t}\right]+V_m e^{j\phi}e^{j5t}=3e^{j5t}$$

$$j10V_m e^{j\phi}+V_m e^{j\phi}=3$$

$$V_m e^{j\phi}=\frac{3}{1+j10}=\frac{3}{\sqrt{1+100}}\ e^{-j\tan^{-1}(10/1)}$$

$$=0.02985\,e^{-j84.3^\circ}\ [V]$$

$$=29.85e^{-j84.3^\circ}\ [mV]$$

Hence,

$$v_c(t)=29.85\cos(5t-84.3^\circ)\ [V]$$

10.4 The phasor

Frequency domain(s-domain) representation

◆ Phasor transformation

time domain representation	$\rightarrow$	freq. domain representation
$\uparrow$		$\downarrow$
solution in time domain	$\leftarrow$	solution in freq. domain

Example) sinusoidal forcing fn. : amplitude and phase

$$v(t) = V_m \cos(wt+\theta) \quad \rightarrow \quad V = V_m e^{j\theta} = V_m \angle \theta$$

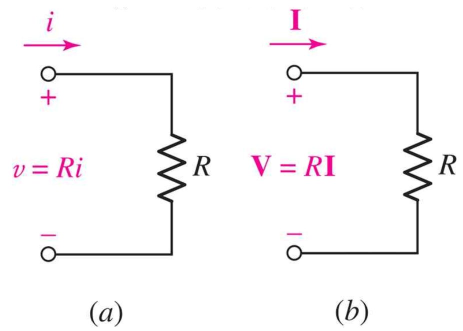
$$i(t) = I_m \cos(wt+\phi) \quad \rightarrow \quad I = I_m e^{j\phi} = I_m \angle \phi$$

Example 10.3) Find phasor representation of $v(t) = 100\cos(400t - 30^{\circ})$.

sol)

$V = 100\angle -30^{\circ}$ at freq. $w = 400$ rad/sec

◆ phasor relationships for R



$$v(t) = V_m\cos(wt+\theta) \rightarrow V_m e^{j(wt+\theta)}$$

$$i(t) = I_m\cos(wt+\phi) \rightarrow I_m e^{j(wt+\phi)}$$

Ohm's Law

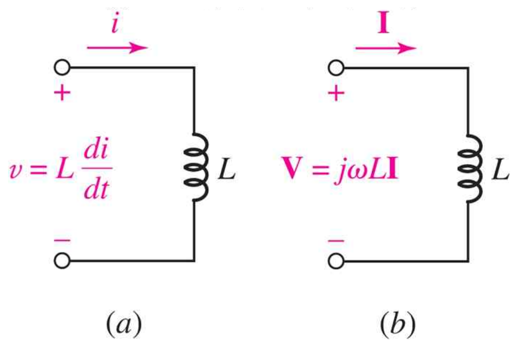
$$v(t) = Ri(t)$$

$$V_m e^{j(wt+\theta)} = RI_m e^{j(wt+\phi)}$$

$$V_m e^{j\theta} = RI_m e^{j\phi}$$

$$V = RI$$

◆ phasor relationships for L



$$v(t) = L\frac{di}{dt}$$

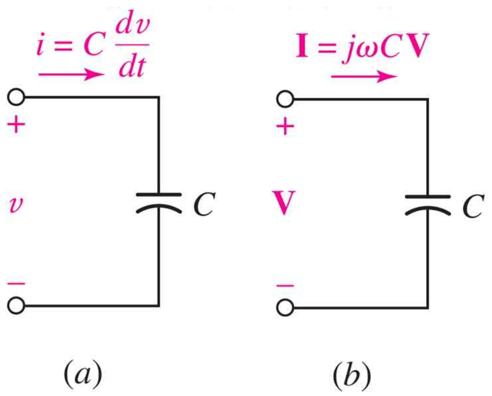
$$V_m e^{j(wt+\theta)} = L\left[I_m e^{j\phi}(jw)e^{jwt}\right]$$

$$V_m e^{j\theta} = jwLI_m e^{j\phi}$$

$$V = jwL\,I$$

◆ phasor relationships for C

$$i(t) = C\frac{dv}{dt}$$
$$I_m e^{j(wt+\phi)} = C\left[V_m e^{j\phi}(jw)e^{jwt}\right]$$
$$I_m e^{j\theta} = jwCV_m e^{j\phi}$$
$$I = jwC\,V$$



◆ Impedance and Admittance

$$Z \equiv \frac{V}{I} \qquad Y \equiv \frac{I}{V} = \frac{1}{Z}$$

: Ohm's law in the freq. domain

(1) Resistor $Z = R$
(2) Inductor $Z = jwL$
(2) capacitor $Z = \frac{1}{jwC}$

Example 10.4) Find $i(t)$ for $L = 4H$ inductor applying $v(t) = 8\cos(100t - 50^\circ)$

sol)

$$V = 8\angle -50^\circ \text{ at } w = 400\ rad/\text{sec}$$
$$I = \frac{V}{Z} = \frac{8\angle -50^\circ}{j400 \times 3} = \frac{8\angle -50^\circ}{1200\angle 90^\circ} = 0.02\angle -140^\circ \ [A]$$

Hence, $i(t) = 0.02\cos(400t - 140^\circ)\ [A]$
$= 20\cos(400t - 140^\circ)\ [mA]$

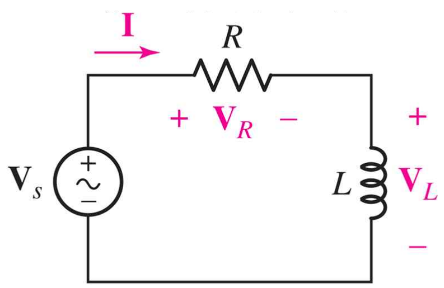
◆ All circuit analysis techniques

the same technique
KVL, KCL, Mesh analysis, Nodal analysis, Thevenin's equi. etc.

Example) Find I, given $V_s = V_m\angle 0^\circ$ $[v_s(t) = V_m\cos(wt)]$

sol)

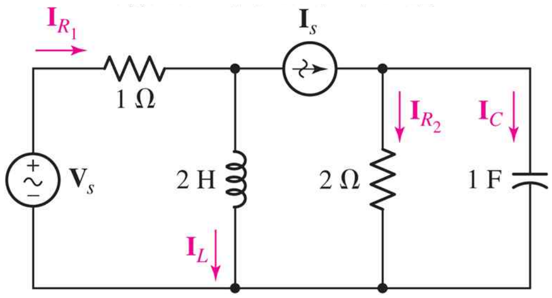
KVL $V_R + V_L = V_s$
$RI + jwLI = V_m\angle 0^\circ$



$$I = \frac{V_m \angle 0^\circ}{R + jwL} = \frac{V_m \angle 0^\circ}{\sqrt{R^2 + w^2 L^2} \angle \tan^{-1}(wL/R)}$$
$$= \frac{V_m}{\sqrt{R^2 + w^2 L^2}} \angle -\tan^{-1}(wL/R)$$

Hence, $i(t) = \dfrac{V_m}{\sqrt{R^2 + w^2 L^2}} \cos(wt + \phi)$ where $\phi = -\tan^{-1}(wL/R)$

Example 10.5) Find I_s and $i_s(t)$ operating at $w = 2\ rad/\text{sec}$, Given $I_c = 2\angle 28^\circ$



sol)

$$V_c = \frac{1}{jwC} \times I_c = \frac{I_c}{j2} = \frac{2\angle 28^\circ}{2\angle 90^\circ} = 1\angle -62^\circ$$

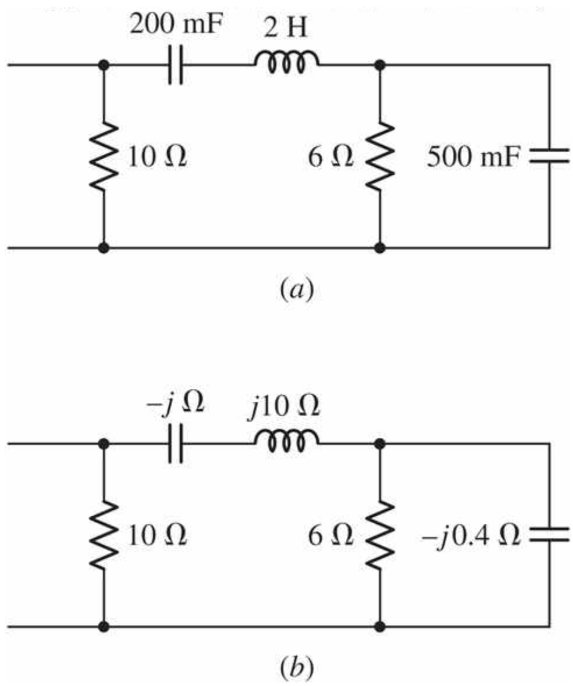
$$I_{R2} = \frac{V_c}{2\Omega} = 0.5\angle -62^\circ$$

$$I_s = I_{R2} + I_c = 0.5\angle -62^\circ + 1\angle -62^\circ = 1.5\angle -62^\circ$$

Hence, $i_s(t) = 1.5\cos(2t - 62^\circ)\ [A]$

Example 10.6) Find equivalent impedance.

$w = 5\ rad/\text{sec}$



sol)

$$200mF \rightarrow Z_c = \frac{1}{jwC} = \frac{1}{j} = -j$$

$$500mF \rightarrow Z_c = \frac{1}{jwC} = \frac{1}{j2.5} = -j0.4$$

$$2H \rightarrow Z_L = jwL = j10$$

$$Z_{eq} = 10\Omega // \{-j + j10 + [6// -j0.4]\}$$
$$= 4.255 + j4.929\ [\Omega]$$

where $6// -j0.4 = \dfrac{-j2.4}{6 - j0.4} = 0.02655 - j0.3982$

Example 10.7) Find $i(t)$

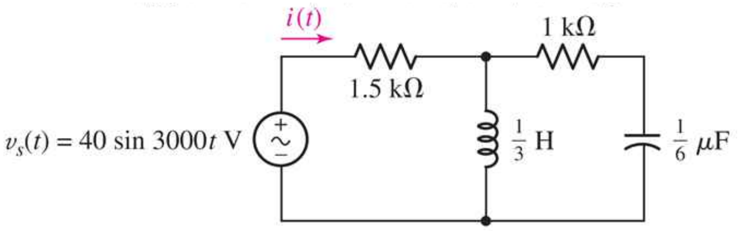
sol)

$w=3{,}000\ rad/\sec$

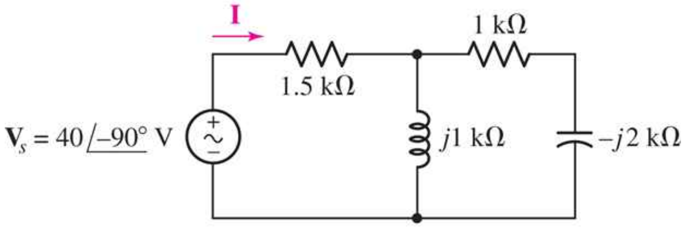
$v_s(t)=40\sin(3{,}000t)$

$\quad =40\cos(3{,}000t-90^\circ)$

$\quad \Rightarrow\ V_s=40\angle-90^\circ$



(a)



(b)

$L=\dfrac{1}{3}\ [H] \qquad \rightarrow\ Z_L=jwL=j1{,}000=j1k$

$C=\dfrac{1}{6}\ [F] \qquad \rightarrow\ Z_C=\dfrac{1}{jwC}=-j2k$

$$=\frac{1}{j3{,}000\times\dfrac{10^{-6}}{6}}$$

$Z_{eq}=1.5+j1//[1k-j2]$

$\quad =2+j1.5\ [k\Omega]=2.5\angle 36.87^\circ\ [k\Omega]$

Phasor Current

$$I=\frac{V_s}{Z_{eq}}=\frac{40\angle-90^\circ}{2.5\angle 36.87^\circ}=16\angle-126.87^\circ\ \ [mA]$$

Hence,

$$i(t)=16\cos(3{,}000t-126.87^\circ)\ [mA]$$

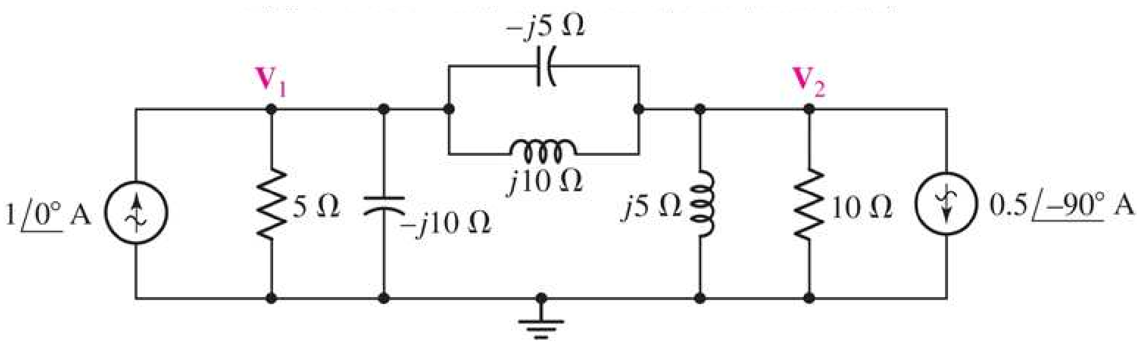
◆ Warning!

$i(t)=\dfrac{40\sin(3{,}000t)}{2.5\angle 36.9^\circ}\ \ (\ \times\ )$

$i(t)\ =\ i(t)=\dfrac{40\sin(3{,}000t)}{2+j1.5}\ \ (\ \times\ )$

10.6 Nodal and Mesh analysis

Example 10.8) Find $v_1(t)$ and $v_2(t)$



sol)

At the left node, KCL

$$\frac{V_1}{5}+\frac{V_1}{-j10}+\frac{V_1-V_2}{-j5}+\frac{V_1-V_2}{j10}=1\angle 0^\circ=1+j0$$

At the right node, KCL

$$\frac{V_2-V_1}{-j5}+\frac{V_2-V_1}{j10}+\frac{V_2}{j5}+\frac{V_2}{10}=0.5\angle -90^\circ=j0.5$$

Multiplying 10 to the eqs.

$$2V_1+jV_1+j2(V_1-V_2)-j(V_1-V_2)=10$$

$$j2(V_2-V_1)-j(V_2-V_1)-j2V_2+V_2=j5$$

Arranging

$$(2+j2)V_1-j\,V_2=10$$

$$-jV_1+(1-j)V_2=j5$$

Crammer's law

$$V_1=\frac{\begin{vmatrix}10 & -j\\ j5 & 1-j\end{vmatrix}}{\begin{vmatrix}2+j2 & -j\\ -j & 1-j\end{vmatrix}}=\frac{10-j10-5}{(2+j2)(1-j)+1}=\frac{5-j10}{5}=1-j2=2.24\angle -63.4^\circ$$

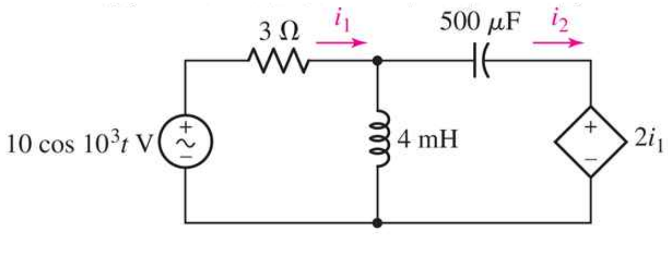
$$V_2=\frac{\begin{vmatrix}2+j2 & 10\\ -j & j5\end{vmatrix}}{\begin{vmatrix}2+j2 & -j\\ -j & 1-j\end{vmatrix}}=\frac{-10+j10-(-j10)}{5}=\frac{-10+j20}{5}=-2+j4=4.47\angle 116.6^\circ$$

Hence,

$$v_1(t)=2.24\cos(wt-63.4^\circ)\ [V]$$

$$v_2(t)=4.47\cos(wt+116.6^\circ)\ [V]$$

Example 10.9) Find $i_1(t)$ and $i_2(t)$



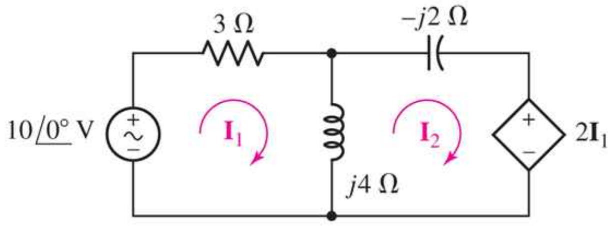
(a)

sol)

$w = 1{,}000\ rad/\sec$

$4mH \ \rightarrow jwL = j4$

$500\mu F \ \rightarrow \dfrac{1}{jwC} = -j2$



(b)

Around left mesh

$$3I_1 + j4(I_1 - I_2) = 10$$

$$\Rightarrow \ (3 + j4)I_1 - j4I_2 = 10 \qquad (*)$$

Around right mesh

$$j4(I_2 - I_1) - j2I_2 + 2I_1 = 0$$

$$\Rightarrow \ (2 - j4)I_1 + j2I_2 = 0 \qquad (**)$$

(*) + 2 ×(**)

$$(7 - j4)I_1 = 10$$

$$\Rightarrow \ I_1 = \frac{10}{7 - j4} = \frac{10}{8.06\angle -29.75} = 1.24\angle 29.75^\circ$$

$$\Rightarrow \ I_2 = 2.77\angle 56.3^\circ$$

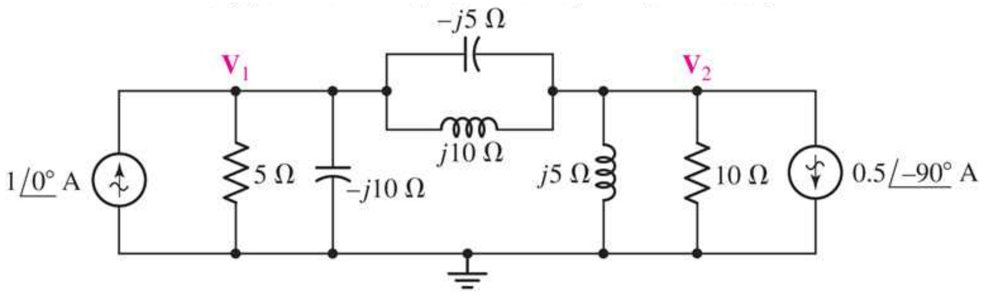
Hence,

$$i_1(t) = 1.24\cos(1{,}000t + 29.7^\circ)\ [A]$$

$$i_2(t) = 2.77\cos(1{,}000t + 56.3^\circ)\ [A]$$

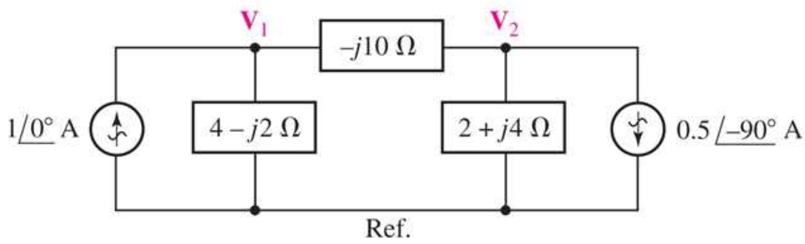
10.7 Superposition, source transformations, and Thevenin's thm

Example 10.10) Find $v_1(t)$ using superposition, Return to Example 10.8).



sol)

Simplification



(1) Only the left source, $I_L = 1A$ (Right source open)

$$V_{1L} = I_{L,4-j2} \times (4-j2)$$
$$= \frac{1-j3}{3-j4} \times (4-j2) = \frac{-2-j14}{3-j4} = 2-j2\ [V]$$

where $I_{L,4-j2} = \dfrac{(2-j6)}{4-j2+(2-j6)} \times I_L = \dfrac{2-j6}{6-j8} \times 1A = \dfrac{1-j3}{3-j4}$

(2) Only the right source, $I_R = 0.5\angle -90^\circ = -j0.5$ (Left source open)

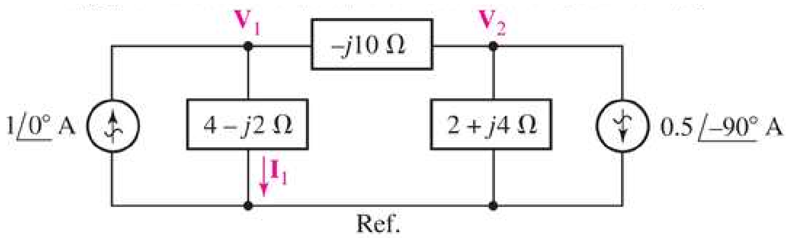
$$V_{1R} = I_{R,4-j2} \times (4-j2)$$
$$= \frac{-2+j}{6-j8} \times (4-j2) = -1\ [V]$$

where $I_{R,4-j2} = \dfrac{2+j4}{2+j4+(4-j12)} \times (-I_R) = \dfrac{2+j4}{6-j8} \times (j0.5) = \dfrac{-2+j}{6-j8}$

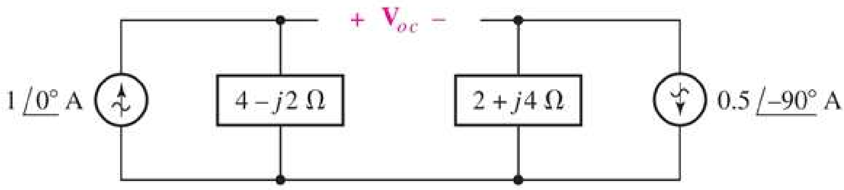
Total current

$$V_1 = V_{1L} + V_{1R} = (2-j2) + (-1) = 1-j2 = 2.24\angle -63.4^\circ$$
$$\Rightarrow\ v_1(t) = 2.24\cos(wt - 63.4^\circ)\ [V]$$

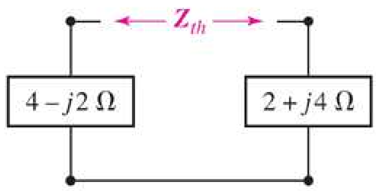
Example 10.11) Find Thevenin's equivalent, faced by the $-j10\ \Omega$, Return to Example 10.10)

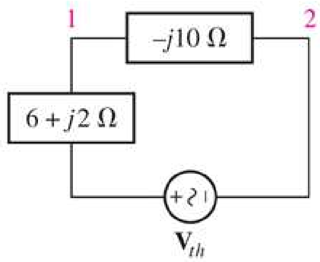


(a)



(b)





(c) (d)

sol)

(1) Open circuit voltage, V_{oc} (Open the impedance $-j10\ \Omega$) ~ Fig. (b)

$$
\begin{aligned}
V_{oc} &= V_{4-j2} - V_{2+j4} \\
&= (4-j2)\times 1A - (2+j4)\times(+j0.5) \\
&= 6-j3\ [V]
\end{aligned}
$$

(2) Equivalent Impedance, Z_{th} (Remove the independent sources) ~ Fig. (c)

$$Z_{th} = (4-j2)+(2+j4) = 6+j2\ [\Omega]$$

Thevenin's equi. circuit ~ Fig. (d)

Current through the impedance, $-j10\ \Omega$

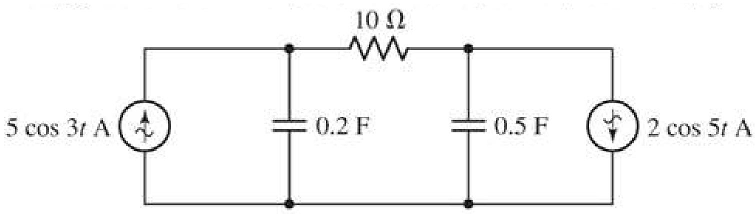
$$I_{12} = \frac{V_{th}}{(6+j2)+(-j10)} = \frac{6-j3}{6-j8} = 0.6+j0.3\ [A]$$

Downward current through the $4-j2\ \Omega$ branch.

$$I_1 = 1A - I_{12} = 1-(0.6+j0.3) = 0.4-j0.3\ [A]$$

$$\Rightarrow\ V_1 = I_1 \times (4-j2) = (0.4-j0.3)\times(4-j2) = 1-j2\ [V]$$

Example 10.12) Find the power $P_{10\Omega}$.

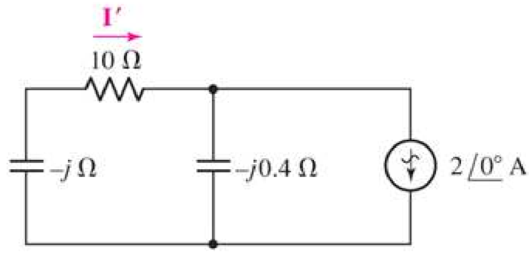


(a)

sol)

2 sources operating different freq.

(1) Right source



(b)

$$w = 5\ rad/\sec$$

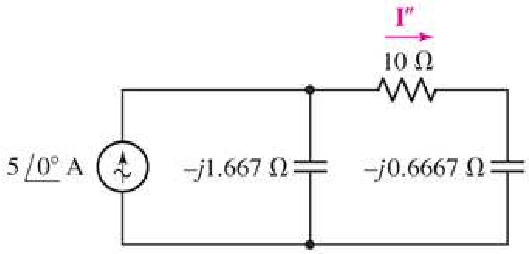
$$0.2F \ \rightarrow \ \frac{1}{jwC} = -j$$

$$0.5F \ \rightarrow \ \frac{1}{jwC} = -j0.4$$

$$\Rightarrow \ I' = \frac{-j0.4}{10-j-j0.4} \times 2A = 79.23\angle -82.03^\circ \ \ [mA]$$

$$\Rightarrow \ i'(t) = 79.23\cos(5t - 82.03^\circ)\ [mA]$$

(2) Left source



(c)

$$w = 3\ rad/\sec$$

$$0.2F \ \rightarrow \ \frac{1}{jwC} = -j1.667$$

$$0.5F \ \rightarrow \ \frac{1}{jwC} = -j0.6667$$

$$\Rightarrow \ I'' = \frac{-j1.667}{10-j0.6667-j1.667} \times 5A = 811.7\angle -78.86^\circ \ \ [mA]$$

$$\Rightarrow \ i'(t) = 811.7\cos(3t - 78.86^\circ)\ [mA]$$

Total current, $i_{10\Omega} = i' + i''$

$$= 79.23\cos(5t - 82.03^\circ) + 811.7\cos(3t - 78.86^\circ)\ [mA]$$

The power

$$p_{10\Omega} = i_{10\Omega}^2 \times 10\Omega$$

$$= 10 \times [79.23\cos(5t - 82.03^\circ) + 811.7\cos(3t - 78.86^\circ)]^2 \ \ [\mu W]$$

<Homeworks>

연습문제 1~9, 10~70 (그림 있는 문제)