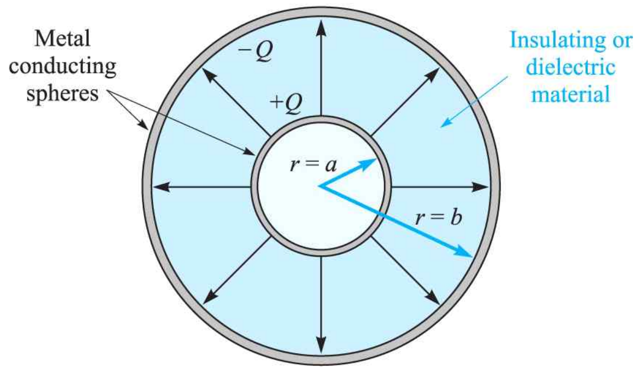
3장. 전속밀도, 가우스의 법칙 및 벡터계의 발산

3.1 전속밀도



• 전하(Q)가 존재하면 주변에

전속(Ψ; electric flux) 발생; $\Psi = Q$ C

• 전속밀도(D; electric flux density);

단위면적을 지나는 전속의 양

$$\rightarrow\ D = \frac{\Psi}{S} = \frac{Q}{4\pi r^2}\boldsymbol{a}_r\ \ \mathrm{C/m^2}$$

• 전계의 세기 $E = \frac{Q}{4\pi\epsilon_0 r^2}\boldsymbol{a}_r$로부터

$$\rightarrow\ D = \epsilon_0 E\ (\text{자유공간})$$

(예제 3.1) z축을 따라 균일한 선전하밀도 8 nC/m → $\rho=3$m에서의 E, D ?

$$E = \frac{\rho_L}{2\pi\epsilon_0\rho}\boldsymbol{a}_\rho = \frac{8\times10^{-9}}{2\pi\cdot(8.854\times10^{-12})\cdot3}\boldsymbol{a}_\rho = 47.9\boldsymbol{a}_\rho\ \ \mathrm{V/m}$$

$$D = \epsilon_0 E = \frac{\rho_L}{2\pi\rho}\boldsymbol{a}_\rho = \frac{8\times10^{-9}}{2\pi\cdot3}\boldsymbol{a}_\rho = 0.424\boldsymbol{a}_\rho\ \ \mathrm{nC/m^2}$$

(ex 1) $Q = 25\mu$C at $O(0,\ 0,\ 0) \rightarrow \Psi$?

(a) $0<\theta<\pi,\ 0<\phi<\frac{\pi}{2},\ r = 20$㎝

$$\Psi = \frac{Q}{4} = 6.25\mu\mathrm{C}$$

(b) $\rho = 8$m, $z = \pm 0.5$m의 폐표면

$$\Psi = Q = 25\mu\mathrm{C}$$

(c) $z = 4$m인 평면

$$\Psi = \frac{Q}{2} = 12.5\mu\mathrm{C}$$

(응용예제 3.1) $Q = 60\mu$C at $O(0,\ 0,\ 0) \rightarrow \Psi$?

(a) $0<\theta<\frac{\pi}{2},\ 0<\phi<\frac{\pi}{2},\ r = 26$㎝의 구의 부분 ⇒ $\Psi = \frac{Q}{8} = 7.5\mu$C

(b) $\rho = 26$㎝, $z = \pm 26$㎝에 의해 정의된 닫힌 표면 ⇒ $\Psi = 60\mu$C

(c) $z = 26$ cm인 평면 ⇒ $\Psi = \frac{Q}{2} = 30\mu$C (ex 2) $\mathbf{D}=?$(직각좌표계로 표시) at $P(6,\ 8,\ -10)$

(a) $Q=30\text{mC}$ at $O(0,\ 0,\ 0)$

$$\mathbf{D}=\epsilon_0\mathbf{E}=\frac{Q}{4\pi r^2}\mathbf{a}_r$$

$$=\frac{Q}{4\pi|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$=\frac{30\times10^{-3}}{4\pi\cdot\left(6^2+8^2+(-10)^2\right)}\cdot\frac{6\mathbf{a}_x+8\mathbf{a}_y-10\mathbf{a}_z}{\sqrt{6^2+8^2+(-10)^2}}$$

$$=5.06\mathbf{a}_x+6.75\mathbf{a}_y-8.44\mathbf{a}_z\ \ \mu\text{C/m}^2$$

(b) z축 위의 균일한 선전하밀도 $\rho_L=40\mu\text{C/m}$

$$\mathbf{D}=\epsilon_0\mathbf{E}=\frac{\rho_L}{2\pi\rho}\mathbf{a}_\rho$$

$$=\frac{\rho_L}{2\pi\sqrt{x^2+y^2}}\frac{x\mathbf{a}_x+y\mathbf{a}_y}{\sqrt{x^2+y^2}}$$

$$=\frac{40\times10^{-6}}{2\pi\cdot\left(6^2+8^2\right)}\cdot\left(6\mathbf{a}_x+8\mathbf{a}_y\right)$$

$$=0.382\mathbf{a}_x+0.509\mathbf{a}_y\ \ \mu\text{C/m}^2$$

(c) $x=9\text{m}$에 있는 평면전하밀도 $\rho_S=\pi/2\ \ \mu\text{C/m}^2$

$$\mathbf{D}=\epsilon_0\mathbf{E}=\frac{\rho_S}{2}(-\mathbf{a}_x)=-\frac{\pi/2\times10^{-6}}{2}\mathbf{a}_x=-0.785\mathbf{a}_x\ \ \mu\text{C/m}^2$$

(응용예제 3.2) $\mathbf{D}=?$(직각좌표계로 표시) at $P(2,\ -3,\ 6)$

(a) $Q=55\text{mC}$ at $O(-2,\ 3,\ -6)$

$$\mathbf{D}=6.38\mathbf{a}_x-9.57\mathbf{a}_y+19.14\mathbf{a}_z\ \ \mu\text{C/m}^2$$

(b) x축 위에 있는 균일한 선전하 $\rho_L=20\text{mC/m}$

$$\mathbf{D}=-212\mathbf{a}_y+424\mathbf{a}_z\ \ \mu\text{C/m}^2$$

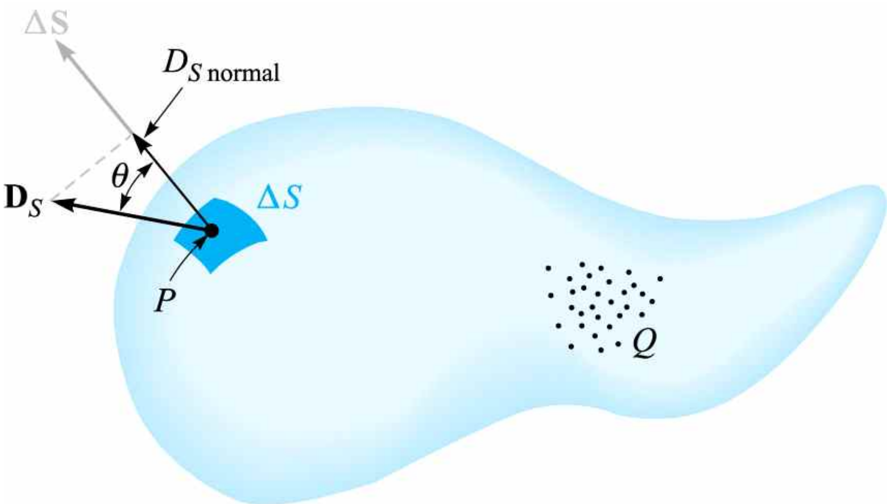
(c) $z=-5\text{m}$ 평면 위에 있는 균일한 표면전하밀도 $\rho_S=120\mu\text{C/m}^2$

$$\mathbf{D}=60\mathbf{a}_z\ \ \mu\text{C/m}^2$$

3.2 가우스 법칙

- 어떤 폐곡면을 통과하는 전속은 그 곡면 내에 있는 총전하량과 같다.

$$\Psi = \int d\Psi = \oint D_S \cdot dS = \text{폐곡면 내의 총전하량} = Q = \int_v \rho_v dv$$



(ex 3) Q at $O(0,\ 0,\ 0)$ → $D = ?$ at $r = a$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

$$D_S = \epsilon_0 E = \frac{Q}{4\pi r^2}\mathbf{a}_r = \frac{Q}{4\pi a^2}\mathbf{a}_r \text{ at } r = a$$

$$dS = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r = a^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$$

$$\oint D_S \cdot dS = \oint \left(\frac{Q}{4\pi a^2}\mathbf{a}_r\right) \cdot \left(a^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r\right)$$

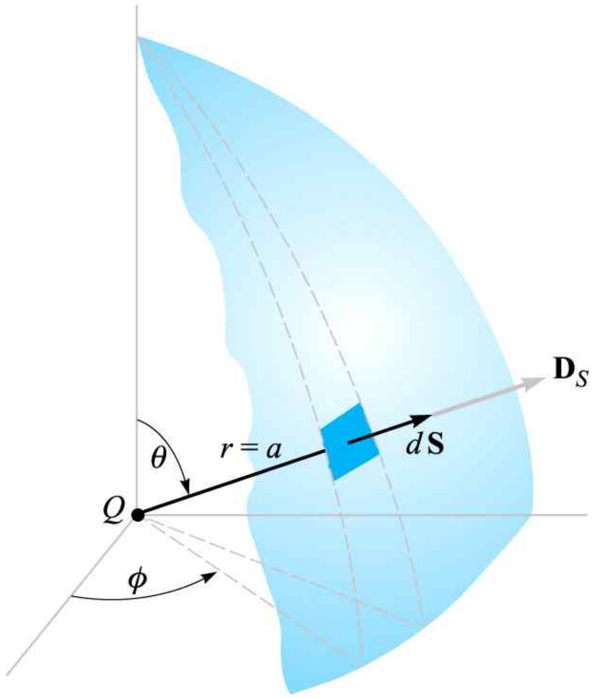
$$= \oint \frac{Q}{4\pi} sin\theta\, d\theta\, d\phi\, \mathbf{a}_r \cdot \mathbf{a}_r$$

$$= \frac{Q}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta\, d\theta\, d\phi$$

$$= \frac{Q}{4\pi}\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta\, d\theta$$

$$= \frac{Q}{4\pi}[\phi]_0^{2\pi}\ [-\cos\theta]_0^{\pi}$$

$$= \frac{Q}{4\pi}[2\pi - 0]\ [-(-1)-(-1)] = Q$$



(ex 4) $\boldsymbol{D} = \frac{r}{3}\boldsymbol{a}_r$ nC/m^2 (자유공간)

(a) $E = ?$ at $r = 0.2$m

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\epsilon_0} = \frac{r \times 10^{-9}}{3\epsilon_0}\boldsymbol{a}_r$$

$$\therefore\ E = \frac{0.2 \times 10^{-9}}{3 \times 8.854 \times 10^{-12}} = 7.53\text{V/m}$$

(b) $Q = ?$ at $r = 0.2$m

$$Q = \Psi = \oint \boldsymbol{D}_S \cdot d\boldsymbol{S} = \oint \frac{r \times 10^{-9}}{3}\boldsymbol{a}_r \cdot r^2 \sin\theta\, d\theta\, d\phi\, \boldsymbol{a}_r$$

$$= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{0.2^3 \times 10^{-9}}{3} sin\theta\, d\theta\, d\phi$$

$$= \int_{\theta=0}^{\theta=\pi} \frac{0.2^3 \times 10^{-9} \times 2\pi}{3} sin\theta\, d\theta$$

$$= \frac{0.2^3 \times 10^{-9} \times 2\pi}{3}(-\cos\theta)_0^{\pi}$$

$$= 33.5\text{pC}$$

(c) $r = 0.3$m인 구를 통과하는 $\Psi = ?$

$$\Psi = \oint \boldsymbol{D}_S \cdot d\boldsymbol{S} = \frac{r \times 10^{-9}}{3} \cdot 4\pi r^2$$

$$= \frac{0.3^3 \times 10^{-9} \times 4\pi}{3}$$

$$= 113.1\text{pC}$$

(응용예제 3.3) $\boldsymbol{D} = 0.3r^2\boldsymbol{a}_r$ nC/m^2 (자유공간)

(a) $\boldsymbol{E} = ?$ at $P(r = 2,\ \theta = 25°, \phi = 90°)$

$\boldsymbol{E} = 135.5\boldsymbol{a}_r$ V/m

(b) $r = 3$인 구 내부에서의 $Q = ?$

$Q = 305$ nC

(c) $r = 4$인 구를 떠나는 총전속선

$\Psi = 965$ nC

(ex 5) $r = 2.5$m인 구의 표면을 지나는 총전속 Ψ?

(a) $Q = 2^{-x^2}$nC at $x = 0,\ \pm1,\ \pm2,\ \ldots,\ \pm m$

$\Psi =$ 폐곡면내의 총전하량

$$= 2^{-x^2}\times10^{-9}\ \Big|_{x=0} + 2^{-x^2}\times10^{-9}\ \Big|_{x=\pm1} + 2^{-x^2}\times10^{-9}\ \Big|_{x=\pm2}$$

$$= (1+2\times2^{-1}+2\times2^{-4})\times10^{-9} = 2.125\text{nC}$$

(b) $\rho_L = \dfrac{1}{z^2+1}$nC/m at z축

$\Psi =$ 폐곡면내의 총전하량 $= Q = \int_v \rho_v dv = \int_l \rho_L dl$

$$= \int_{z=-2.5}^{z=2.5} \frac{1\times10^{-9}}{z^2+1}dz = \ldots = 2.38\text{nC}$$

(c) $\rho_S = \dfrac{1}{x^2+y^2+4}$nC/m^2 at $z=0$

$\Psi =$ 폐곡면내의 총전하량 $= Q = \int_v \rho_v dv = \int_S \rho_S dS$

$$= \int_{\phi=0}^{\phi=2\pi}\int_{r=0}^{r=2.5} \frac{1\times10^{-9}}{x^2+y^2+4} r dr d\phi = 2\pi\int_{r=0}^{r=2.5}\frac{r\times10^{-9}}{r^2+4}dr$$

$$= \ldots = 2.96\text{nC}$$

(응용예제 3.4) $x,\ y,\ z = \pm5$에 의해 형성된 육면체의 표면을 떠나는 총전속 Ψ?

(a) $Q_1 = 0.1\mu$C at $P_1(1,\ -2,\ 3)$ & $Q_2 = \dfrac{1}{7}\mu$C at $P_2(-1,\ 2,\ -2)$

$\Psi = 0.243\mu$C

(b) $\rho_L = \pi\mu$C/m at $x = -2,\ y = 3$

$\Psi = 31.4\mu$C

(c) $\rho_S = 0.1\mu$C/m^2 at $y = 3x$

$\Psi = 10.54\mu$C

3.3 가우스 법칙의 응용예; 대칭전하 분포

- $Q = \oint \mathbf{D}_S \cdot d\mathbf{S} \quad \rightarrow \quad D_S = ?$
- 가우스 표면의 정의

  1. 모든 점에서 $\mathbf{D}_S$가 폐곡면과 수직이거나 접선방향

     $\rightarrow \mathbf{D}_S \cdot d\mathbf{S} = D_S dS$ 또는 $\mathbf{D}_S \cdot d\mathbf{S} = 0$

  2. $\mathbf{D}_S \cdot d\mathbf{S} = D_S dS$인 곳에서는 $\int D_S dS = D_S \int dS = D_S \Delta S$를 만족

(ex 6) 원점에 점전하 $Q \rightarrow r$만큼 떨어진 곳에서의 D_S?

(a) $E = \dfrac{Q}{4\pi\epsilon_0 r^2} \quad \rightarrow \quad D_S = \epsilon_0 E = \dfrac{Q}{4\pi r^2}$

(b) $Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \oint_{sph} D_S dS = D_S \oint_{sph} dS$

$= D_S \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} r^2 \sin\theta d\theta d\phi$

$= 4\pi r^2 D_S \qquad \rightarrow D_S = \dfrac{Q}{4\pi r^2}$

(ex 7) 길이가 무한대인 선전하(ρ_L C/m)에 의한 전속밀도 D_S

길이가 L m인 원통 모양의 가우스 표면을 잡아주면

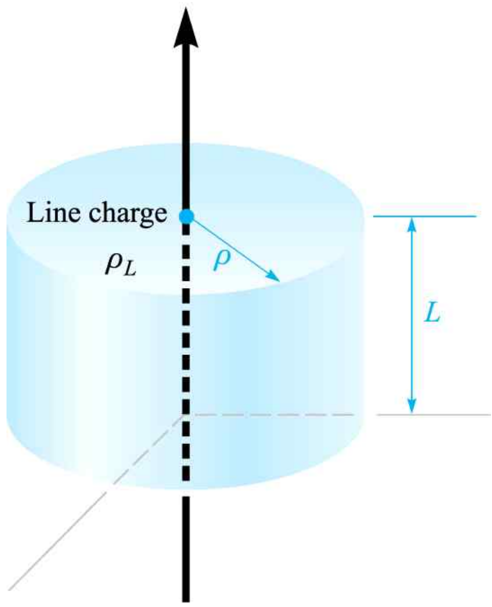
$Q = \oint_{cyl} \mathbf{D}_S \cdot d\mathbf{S}$

$= D_S \int_{sides} dS + 0 \int_{top} dS + 0 \int_{bottom} dS$

$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho d\phi dz$

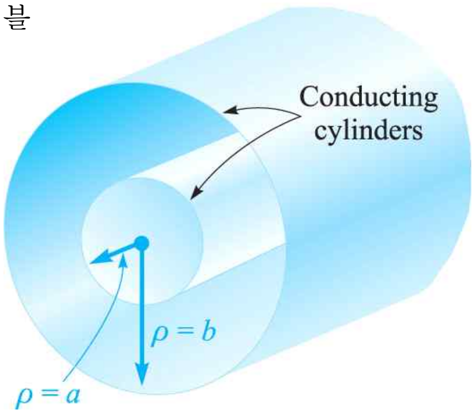
$= D_S \; 2\pi\rho L$

$\therefore \; D_S = D_\rho = \dfrac{Q}{2\pi\rho L} = \dfrac{\rho_L L}{2\pi\rho L} = \dfrac{\rho_L}{2\pi\rho}$



(예제 3.2) $a = 1\text{mm}$, $b = 4\text{mm}$, $L = 50\text{cm}$, $Q_1 = 30\text{nC}$인 동축 케이블

$\rightarrow \rho_{S_1}$, ρ_{S_2}, $\boldsymbol{E}$, $\boldsymbol{D} = ?$



ρ_{S_1}; $\quad \rho_{S_1} = \dfrac{Q_1}{S_1} = \dfrac{Q_1}{2\pi aL} = \dfrac{30\times10^{-9}}{2\pi\times10^{-3}\times0.5} = 9.55\mu\text{C/m}^2$

ρ_{S_2}; $\quad Q_2 = -Q_1 = -30\text{nC}$

$$\rho_{S_2} = \frac{Q_2}{S_2} = \frac{-Q_1}{2\pi bL} = \frac{-30\times10^{-9}}{2\pi\times(4\times10^{-3})\times0.5} = -2.39\mu\text{C/m}^2$$

$\boldsymbol{D}$; $a \leqq \rho < b$일 때

$$Q_1 = \oint_{cyl} \boldsymbol{D_s}\cdot d\boldsymbol{S} = 0\int_{left} dS + D_\rho \int_{sides} dS + 0\int_{right} dS = D_\rho 2\pi\rho L$$

$$D_\rho = \frac{Q_1}{2\pi\rho L} = \frac{\rho_{S_1} 2\pi aL}{2\pi\rho L} = \frac{a\rho_{S_1}}{\rho} = \frac{10^{-3}\times(9.55\times10^{-6}}{\rho} = \frac{9.55}{\rho}\text{nC/m}^2$$

$\boldsymbol{E}$; $a \leqq \rho < b$일 때

$$E_\rho = \epsilon_0 D_\rho = \frac{9.55\times10^{-9}}{8.854\times10^{-12}\rho} = \frac{1079}{\rho}\text{V/m}$$

(ex 8) $\quad 200\mu\text{C/m}^2$ at $r = 3\text{cm}$

$-50\mu\text{C/m}^2$ at $r = 5\text{cm}$ $\quad\rightarrow\quad \boldsymbol{D} = ?$

$\rho_{S_x}\mu\text{C/m}^2$ at $r = 7\text{cm}$

(a) at $r = 2\text{cm}$

$Q_{enc} = 0 \rightarrow D_S = 0$

(b) at $r = 4\text{cm}$

$$Q = D_S \int_{sph} dS = D_S 4\pi r^2$$

$$D_S = D_r = \frac{Q}{4\pi r^2} = 200\times10^{-6}\times\frac{4\pi\times0.03^2}{4\pi\times0.04^2} = 112.5\mu\text{C/m}^2$$

(c) at $r = 6\text{cm}$

$$Q = D_S \int_{sph} dS = D_S 4\pi r^2$$

$$D_S = \frac{Q}{4\pi r^2} = \frac{Q_1 + Q_2}{4\pi r^2} = \frac{(200\times10^{-6})(4\pi\times0.03^2) + (-50\times10^{-6})(4\pi\times0.05^2)}{4\pi\times0.06^2}$$

$$= \frac{200\times0.03^2 - 50\times0.05^2}{0.06^2}\times10^{-6}$$

$$= 15.28\mu\text{C/m}^2$$

(d) $r=7.32$㎝일 때 $\boldsymbol{D}=0 \quad \rightarrow \quad \rho_{S_x}=?$

$$Q=D_S\int_{sph} dS=D_S 4\pi r^2$$

$$D_S=\frac{Q}{4pr^2}=\frac{Q_1+Q_2+Q_3}{4\pi\times 0.0732^2}=0$$

$$200\times 0.03^2-50\times 0.05^2+\rho_{S_x}\times 0.07^2=0$$

$$\rho_{S_x}=\frac{200\times 0.03^2-50\times 0.05^2}{-0.07^2}=-11.22\mu\text{C/㎡}$$

(응용예제 3.5) 0.25μC at $r=0$

2mC/㎡ at $r=1$㎝ $\quad \rightarrow \quad \boldsymbol{D}=?$

-0.6mC/㎡ at $r=1.8$㎝

(a) at $r=0.5$㎝

(b) at $r=1.5$㎝

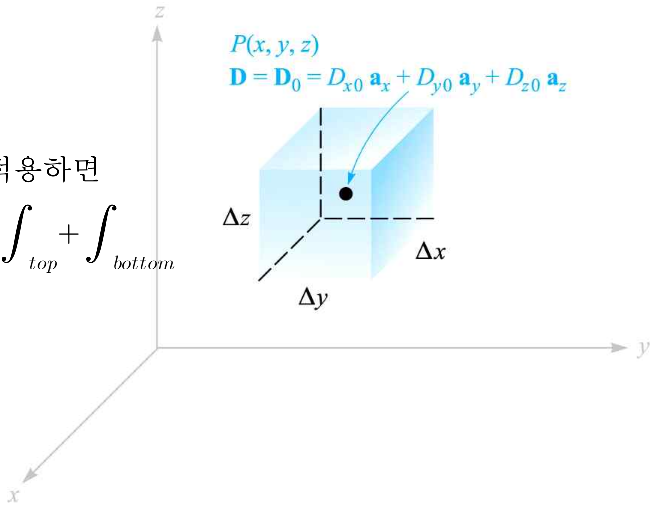
(c) at $r=2.5$㎝

(d) $r=3.5$㎝에서 $\boldsymbol{D}=0 \quad \rightarrow \quad \rho_{S_x}=?$ at $r=3$㎝

3.4 가우스 법칙의 응용예; 미소체적소

- 가우스 표면을 미소체적소로 정의 → 가우스 표면에 대한 일반화

$$\oint_S D \cdot dS = Q$$



직교좌표계에서의 미소체적소에 대하여 적용하면

$$\oint_S D \cdot dS = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

$$\int_{front} \doteq D_{front} \cdot \Delta S_{front}$$

$$\doteq D_{front} \cdot \Delta y \Delta z \boldsymbol{a_x}$$

$$\doteq D_{x,front} \;\Delta y \Delta z$$

$$\leftarrow D_{x,front} \doteq D_{x0} + \frac{\Delta x}{2} \times (x\text{에 대한 } D_x\text{의 변화율})$$

$$\doteq D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

$$\int_{front} \doteq \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{back} \doteq \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\rightarrow \int_{front} + \int_{back} \doteq \frac{\partial D_x}{\partial x}\Delta x \Delta y \Delta z$$

$$\therefore \oint_S D \cdot dS = Q \doteq \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}\right)\Delta x \Delta y \Delta z$$

$$= \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}\right)\Delta v$$

$$\rightarrow \quad \Delta v\text{내의 전하량} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times (\text{체적 } \Delta v)$$

$$\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

(ex 9) $\boldsymbol{D} = y^2z^3\boldsymbol{a}_x + 3xyz^3\boldsymbol{a}_y + 3xy^2z^2\boldsymbol{a}_z$ pC/㎡ (자유공간)

(a) 원점으로부터 $x = 3,\ 0 \leqq y \leqq 2,\ 0 \leqq z \leqq 1$인 표면을 통과하는 총전속

$$\Psi = \int_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_S D_x dydz$$

$$= \int_{z=0}^{z=1}\int_{y=0}^{y=2} y^2z^3 \times 10^{-12} dydz = \left[\frac{1}{3}y^3\right]_{y=0}^{y=2} \times \left[\frac{1}{4}z^4\right]_{z=0}^{z=1} \times 10^{-12} = 0.667\text{pC}$$

(b) $\boldsymbol{E}$ at $P(3,\ 2,\ 1)$

$$\boldsymbol{D}\ |_{P(3,\ 2,\ 1)} = 4\boldsymbol{a}_x + 18\boldsymbol{a}_y + 36\boldsymbol{a}_z \ \text{pC/㎡}$$

$$\therefore\ \boldsymbol{E} = \frac{\boldsymbol{D}}{\epsilon_0} = \frac{4\boldsymbol{a}_x + 18\boldsymbol{a}_y + 36\boldsymbol{a}_z}{8.854}\ \text{V/m}$$

(c) 점 $P(3,\ 2,\ 1)$에서 반경 2μm의 미소구에 포함된 총전하

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = Q \doteqdot \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}\right)\Delta v$$

$$= \left(0 + 3xz^3 + 6xy^2z\right)\ |_{P(3,\ 2,\ 1)} \times 10^{-12} \times \frac{4}{3}\pi \cdot (2\times 10^{-6})^3$$

$$= (9 + 72) \times \frac{4}{3}\pi \times 8 \times 10^{-30} = 2.71 \times 10^{-27}\ \text{C}$$

(예제 3.3) $\boldsymbol{D} = e^{-x}\sin y\boldsymbol{a}_x - e^{-x}\cos y\boldsymbol{a}_y + 3z\boldsymbol{a}_z$ C/㎡

$\Delta v = 10^{-9}$㎥ 내의 총 전하량 $Q = ?$

$$Q = \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}\right)\Delta v = (-e^{-x}\sin y + e^{-x}\sin y + 2)\Delta v = 2\text{nC}$$

(응용예제 3.6) $\boldsymbol{D} = 8xyz^4\boldsymbol{a}_x + 4x^2z^4\boldsymbol{a}_y + 16x^2yz^3\boldsymbol{a}_z$ pC/㎡

(a) $z = 2,\ 0 < x < 2,\ 1 < y < 3$인 표면을 $\boldsymbol{a}_z$으로 통과하는 총 전속

$$\Psi = 1365\text{pC}$$

(b) $\boldsymbol{E}$ at $P(2,\ -1,\ 3)$

$$\boldsymbol{D}\ |_{P(2,\ -1,\ 3)} =$$

$$\boldsymbol{E} = \frac{\boldsymbol{D}}{\epsilon_0} = -146.4\boldsymbol{a}_x + 146.4\boldsymbol{a}_y - 195.2\boldsymbol{a}_z\ \text{V/m}$$

(c) 점 $P(2,\ -1,\ 3)$에서 부피 10^{-12}㎥의 미소구에 포함된 총 전하량의 근사치

$$Q = \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}\right)\Delta v$$

$$= (8yz^4 + 0 + 48x^2yz^2) \times 10^{-12}\ |_{P(2,\ -1,\ 3)}\Delta v$$

$$= (-8\times 81 - 48 \times 4 \times 9) \times 10^{-12} \times 10^{-12} = -2.376 \times 10^{-21}\text{C}$$

3.5 벡터계의 발산

- 발산(divergence)의 정의

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = Q \doteq \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} \right) \Delta v$$

$$\left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} \right) \doteq \frac{\oint_S \boldsymbol{D} \cdot d\boldsymbol{S}}{\Delta v} = \frac{Q}{\Delta v}$$

$$\left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \boldsymbol{D} \cdot d\boldsymbol{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$$

임의의 벡터 $\boldsymbol{A}$에 대하여

$$\boldsymbol{A}\text{의 발산} = \mathrm{div}\boldsymbol{A} = \left(\frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint_S \boldsymbol{A} \cdot d\boldsymbol{S}}{\Delta v}$$

- 발산의 물리적 의미

⑴ 전속밀도의 발산은 미소체적소의 폐곡면으로부터 밖으로 나오는 단위체적당 전속의 극한값과 같다.

⑵ 전속밀도의 발산이 0이 아니면 내부에 전하(source)가 존재하고, 0이면 내부에 전하가 존재하지 않는다.

- 각 좌표계에서 발산의 계산

⑴ 직각좌표계

$$\mathrm{div}\boldsymbol{D} = \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}$$

⑵ 원통좌표계

$$\mathrm{div}\boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

⑶ 구좌표계

$$\mathrm{div}\boldsymbol{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

(예제 3.4) $\boldsymbol{D}=e^{-x}\sin y\boldsymbol{a_x}-e^{-x}\cos y\boldsymbol{a_y}+2z\boldsymbol{a_z}$ → $\text{div}\boldsymbol{D}=?$ at $O(0,\ 0,\ 0)$

$$\text{div}\boldsymbol{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}$$

$$=-e^{-x}\sin y+e^{-x}\sin y+2\ \Big|_{O(0,\ 0,\ 0)}=2\text{C/m}^3$$

(ex 10) $\text{div}\boldsymbol{D}=?$

(a) $\boldsymbol{D}=20xy^2(z-1)\boldsymbol{a_x}+20x^2y(z+1)\boldsymbol{a_y}+10x^2y^2\boldsymbol{a_z}$ C/m² at $P_A(0.3,\ 0.4,\ 0.5)$

$$\text{div}\boldsymbol{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}$$

$$=20y^2(z-1)+20x^2(z+1)\ \Big|_{P_A(0.3,\ 0.4,\ 0.5)}$$

$$=20\times0.4^2\times(0.5-1)+20\times0.3^2\times(0.5+1)\ =1.1\ \text{C/m}^3$$

(b) $\boldsymbol{D}=4\rho z\sin\phi\boldsymbol{a_\rho}+2\rho z\cos\phi\boldsymbol{a_\phi}+2\rho^2\sin\phi\boldsymbol{a_z}$ C/m² at $P_H(1,\ \pi/2,\ 2)$

$$\text{div}\boldsymbol{D}=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho)+\frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi}+\frac{\partial D_z}{\partial z}$$

$$=\frac{1}{\rho}8\rho z\sin\phi+\frac{1}{\rho}2\rho z\,(-\sin\phi)\ \Big|_{P_H(1,\ \pi/2,\ 2)}$$

$$=\frac{1}{1}\times8\times1\times2\times\sin\frac{\pi}{2}-\frac{1}{1}\times2\times1\times2\times\sin\frac{\pi}{2}\ =12\text{C/m}^3$$

(c) $\boldsymbol{D}=\sin\theta\cos\phi\boldsymbol{a_r}+\cos\theta\cos\phi\boldsymbol{a_\theta}-\sin\phi\boldsymbol{a_\phi}$ C/m² at $P_C(2,\ \theta=\pi/3,\ \phi=\pi/6)$

$$\text{div}\boldsymbol{D}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2D_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta)+\frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$=\frac{1}{r^2}2r\sin\theta\cos\phi+\frac{1}{r\sin\theta}(\cos^2\theta-\sin^2\theta)\cos\phi-\frac{1}{r\sin\theta}cos\phi$$

$$=\frac{\cos\phi}{r\sin\theta}(2\sin^2\theta+\cos^2\theta-\sin^2\theta-1)$$

$$=\frac{\cos\phi}{r\sin\theta}(\sin^2\theta+\cos^2\theta-1)\ =0$$

(응용예제 3.7) $\text{div}\boldsymbol{D}=?$

(a) $\boldsymbol{D}=(2xyz-y^2)\boldsymbol{a_x}+(x^2z-2xy)\boldsymbol{a_y}+x^2y\boldsymbol{a_z}$ at $P_A(2,\ 3,\ -1)$

(b) $\boldsymbol{D}=2\rho z^2\sin^2\phi\boldsymbol{a_\rho}+\rho z^2\sin2\phi\boldsymbol{a_\phi}+2\rho^2z\sin^2\phi\boldsymbol{a_z}$ at $P_B(2,\ 110°,\ -1)$

(c) $\boldsymbol{D}=2r\sin\theta\cos\phi\boldsymbol{a_r}+r\cos\theta\cos\phi\boldsymbol{a_\theta}-r\sin\phi\boldsymbol{a_\phi}$ at $P_C(1.5,\ 30°,\ 50°)$

(a) -10.00 (b) 9.06 (c) 2.18

3.6 맥스웰의 제1방정식(정전계)

- $\text{div}\boldsymbol{D} = \left(\frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \boldsymbol{A} \cdot d\boldsymbol{S}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$

(ex 11) $\rho_v = ?$

(a) $\boldsymbol{D} = xy^2\boldsymbol{a_x} + yx^2\boldsymbol{a_y} + z\boldsymbol{a_z}$ C/㎡

$$\rho_v = \text{div}\boldsymbol{D} = \frac{\partial Dx}{\partial x} + \frac{\partial Dy}{\partial y} + \frac{\partial Dz}{\partial z}$$

$$= y^2 + x^2 + 1 \ \text{C/㎥}$$

(b) $\boldsymbol{D} = \rho z^2 \sin^2\phi \boldsymbol{a_\rho} + \rho z^2 \sin\phi\cos\phi \boldsymbol{a_\phi} + \rho^2 z \sin^2\phi \boldsymbol{a_z}$ C/㎡

$$\rho_v = \text{div}\boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$= \frac{1}{\rho} 2\rho z^2 \sin^2\phi + \frac{1}{\rho}\rho z^2(\cos^2\phi - \sin^2\phi) + \rho^2 \sin^2\phi$$

$$= z^2(\sin^2\phi + \cos^2\phi) + \rho^2\sin^2\phi$$

$$= z^2 + \rho^2 \sin^2\phi \ \text{C/㎥}$$

(c) $\boldsymbol{D} = \boldsymbol{a_r}$ C/㎡

$$\rho_v = \text{div}\boldsymbol{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

$$= \frac{1}{r^2} 2r$$

$$= \frac{2}{r} \ \text{C/㎥}$$

(응용예제 3.8) $\rho_v = ?$

(a) $\boldsymbol{D} = \frac{4xy}{z}\boldsymbol{a_x} + \frac{2x^2}{z}\boldsymbol{a_y} - \frac{2x^2 y}{z^2}\boldsymbol{a_z}$

(b) $\boldsymbol{D} = z\sin\phi \boldsymbol{a_\rho} + z\cos\phi \boldsymbol{a_\phi} + \rho\sin\phi \boldsymbol{a_z}$

(c) $\boldsymbol{D} = \sin\theta\sin\phi \boldsymbol{a_r} + \cos\theta\sin\phi \boldsymbol{a_\theta} + \cos\phi \boldsymbol{a_\phi}$

(a) $\frac{4y}{z^3}(x^2 + z^2)$

(b) 0

(c) 0

3.7 벡터연산자와 발산정리

- 직각좌표계에서 벡터연산자 ▽(Del)의 정의

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\boldsymbol{a}_z$$

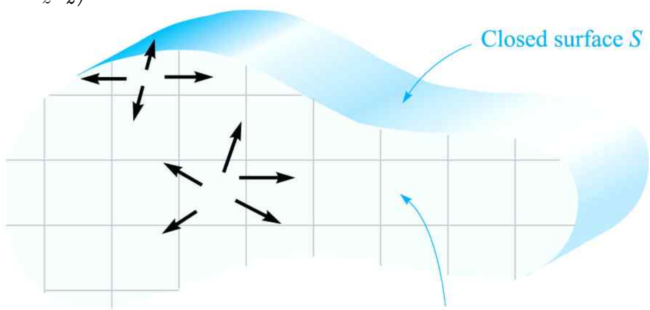
- ▽ 연산자의 적용

$$\nabla \cdot \boldsymbol{D} = \left(\frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\right) \cdot (D_x\boldsymbol{a}_x + D_y\boldsymbol{a}_y + D_z\boldsymbol{a}_z)$$

$$= \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= \mathrm{div}\boldsymbol{D}$$



- 발산정리(divergence theorem)

$$\oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = Q = \int_{vol} \rho_v \, dv = \int_{vol} \nabla \cdot \boldsymbol{D} \, dv$$

임의의 벡터계에서 폐곡면 전체에 대한 벡터의 법선성분의 면적적분은, 폐곡면 내의 체적전체에 대한 이 벡터의 발산의 체적적분과 같다.

(예제 3.5) $\boldsymbol{D} = 2xy\boldsymbol{a}_x + x^2\boldsymbol{a}_y$인 전계 내에 있는 $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$인 6 개 면으로 이루어진 직육면체 내의 전하량을 발산정리를 이용하여 비교하여라.

(a) From Gauss' Law

$$Q = \oint_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_0^3\int_0^2 (\boldsymbol{D})_{x=0} \cdot (-dy\,dz\boldsymbol{a}_x) + \int_0^3\int_0^2 (\boldsymbol{D})_{x=1} \cdot (dy\,dz\boldsymbol{a}_x)$$

$$+ \int_0^3\int_0^1 (\boldsymbol{D})_{y=0} \cdot (-dx\,dz\boldsymbol{a}_y) + \int_0^3\int_0^1 (\boldsymbol{D})_{y=2} \cdot (dx\,dz\boldsymbol{a}_y)$$

$$+ \int_0^2\int_0^1 (\boldsymbol{D})_{z=0} \cdot (-dx\,dy\boldsymbol{a}_z) + \int_0^2\int_0^1 (\boldsymbol{D})_{z=3} \cdot (dx\,dy\boldsymbol{a}_z)$$

$$= -\int_0^3\int_0^2 (D_x)_{x=0}\,dy\,dz + \int_0^3\int_0^2 (D_x)_{x=1}\,dy\,dz$$

$$- \int_0^3\int_0^1 (D_y)_{y=0}\,dx\,dz + \int_0^3\int_0^1 (D_y)_{y=2}\,dx\,dz$$

$$= \int_0^3\int_0^2 (D_x)_{x=1}\,dy\,dz$$

$$= \int_0^3\int_0^2 2y\,dy\,dz = \int_0^3 4\,dz = 12$$

(b) From divergence theorem

$$\nabla \cdot \boldsymbol{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

$$\therefore \int_{vol} \nabla \cdot \boldsymbol{D}\, dv = \int_0^3 \int_0^2 \int_0^1 2y\, dx\, dy\, dz$$

$$= \int_0^3 \int_0^2 2y\, dy\, dz$$

$$= \int_0^3 4\, dz$$

$$= 12$$