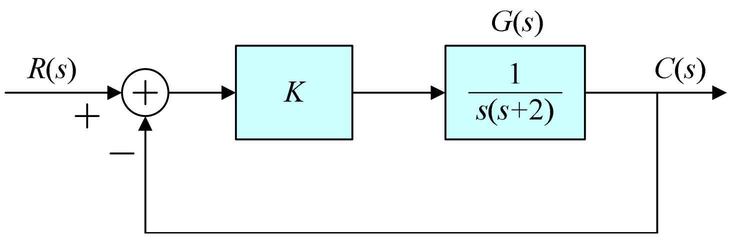
# Chapter 7. Root Locus Method.

The root locus is the trajectory of the roots of the char. eq. in the s−plane as a system parameter varies.

⇒ Graphical information

## 7.2 Root Locus Concept

*Example*) Unity f/b control system - Fig. 7.2




$$T(s)=\frac{\sum_{k=1}^{N} P_k \Delta_k}{\Delta(s)} \quad \text{where} \quad \Delta(s)=1+KG(s)=1+K\frac{1}{s(s+2)}$$

char. eq.

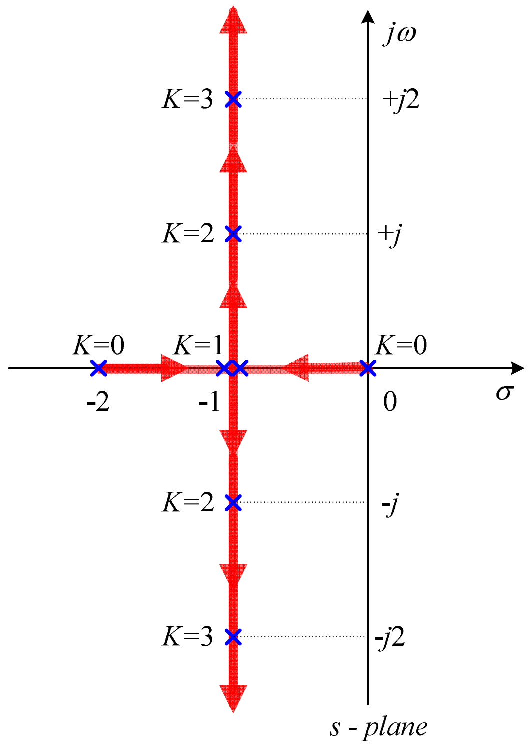
$s^2+2s+K=0$ where $K$ : system parameter ($K = 0 \rightarrow \infty$)

If $K=0$, then $s_{1,2} = 0, -2$

If $K=1$, then $s_{1,2} = -1$

If $K=2$, then $s_{1,2} = -1 \pm j$

If $K=3$, then $s_{1,2} = -1 \pm j2$




※ Root Locus

$K=1 \rightarrow \infty$ (positive real)일 때

특성방정식 $1+KG(s)=0$을 만족하는 $s$점

(특성방정식의 근의 집합)

char. eq.

$$1+KG(s)=0 \quad \Rightarrow \quad KG(s) = -1+j0 \text{ at cartesian coordinate}$$

$$\Rightarrow \quad Ae^{j\Phi} = 1e^{\pm j180^\circ} \text{ at polar coordinate}$$

1) Magnitude Criterion

$$A = Mag.\ of\ KG(s) = |KG(s)| = 1$$

2) Phase Criterion

$$\Phi = Phase\ of\ KG(s) = \measuredangle\{KG(s)\} = \measuredangle\{G(s)\} = \pm 180^\circ$$

The root locus is constructed by finding all points in the s−plane that satisfy the phase criterion, and then the values of $K$ along the locus is determined by the magnitude criterion.

※ Return to the previous example – Fig. 7.2

char. eq. $\quad 1+Kp(s) = 0$

$$s^2+2s+K=0 \quad \Rightarrow \quad 1+K\frac{1}{s(s+2)} = 0 \quad \text{where } p(s) = \frac{1}{s(s+2)}$$

< Phase Criterion >

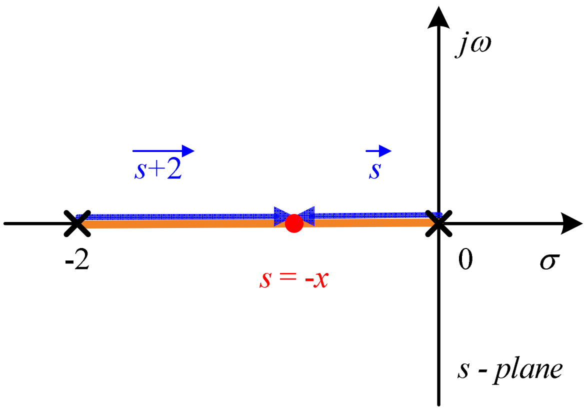
$$\measuredangle\left(\frac{1}{s(s+2)}\right) = \pm 180^\circ \quad \Rightarrow \quad \measuredangle(1) - [\measuredangle(s) + \measuredangle(s+2)] = \pm 180^\circ$$

$$\measuredangle(s) + \measuredangle(s+2) = \pm 180^\circ$$

ex) $s = -x$ , where $x = [-2, 0]$

$\measuredangle(s) = \pm 180^\circ$ and $\measuredangle(s+2) = 0^\circ$

$\therefore)\ \measuredangle(s) + \measuredangle(s+2) = \pm 180^\circ$ : satisfied




ex) $s = -1+jy$, where $y = [-\infty, \infty]$

$\measuredangle(s) + \measuredangle(s+2) = \pm 180^\circ$ : satisfied

at $y=1$

$\measuredangle(s) = 135^\circ$ and $\measuredangle(s+2) = 45^\circ$

ex) the other point

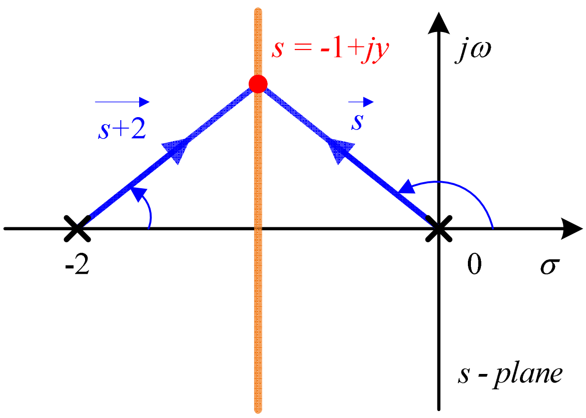
$\measuredangle(s) + \measuredangle(s+2) \neq \pm 180^\circ$

< Magnitude Criterion >

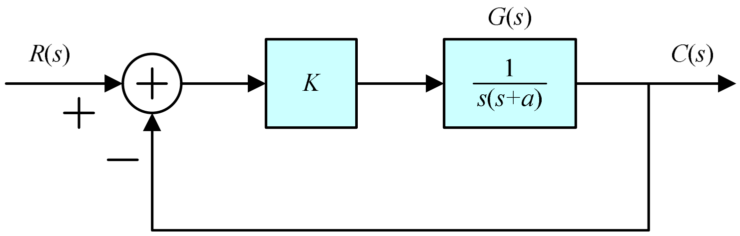
$K=?$ at $s=-1$

sol) $|KG(s)| = \left|\frac{K}{s(s+2)}\right| = \frac{K}{|s||s+2|} = 1$

at $s=-1$, $K = |s||s+2| = 1 \times 1 = 1$



*Example*) Single−loop system - Fig. 7.2




sol) char. eq.

$$1+KG(s)=0 \quad \Rightarrow \quad 1+K\frac{1}{s(s+a)}=0 \quad \Rightarrow \quad s^2+as+K=0 \quad \text{where } a \text{ is parameter}$$

$$1+a\frac{s}{s^2+K}=0 \quad \text{where } \; G(s)=\frac{s}{s^2+K}=\frac{s}{(s+j\sqrt{K})(s-j\sqrt{K})}$$

< Phase Criterion >

$$\measuredangle\left(\frac{s}{s^2+K}\right)=\pm 180^\circ \quad \Rightarrow \quad \measuredangle(s)-\left[\measuredangle(s+j\sqrt{K})+\measuredangle(s-j\sqrt{K})\right]=\pm 180^\circ$$

ex) $s=-x$ on the negative real axis

$\measuredangle(s)=\pm 180^\circ$ and $\measuredangle(s+j\sqrt{K})+\measuredangle(s-j\sqrt{K})=0^\circ$

Hence, phase criterion is satisfied.

ex) $s=-x\pm jy$ on the semicircle

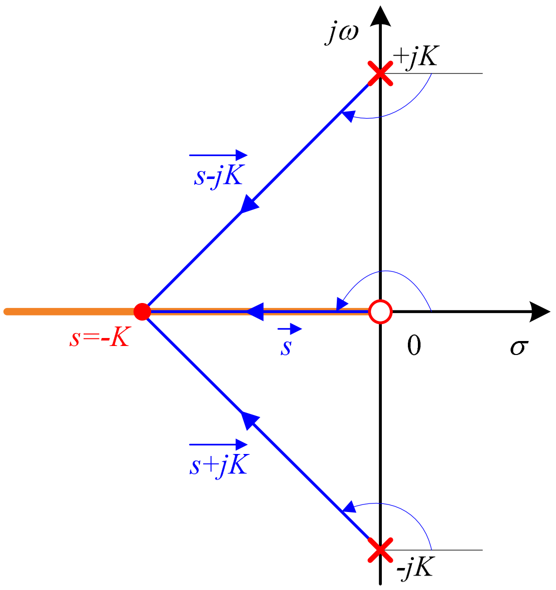
$\measuredangle(s)=A$ and $\measuredangle(s+j\sqrt{K})+\measuredangle(s-j\sqrt{K})=B$

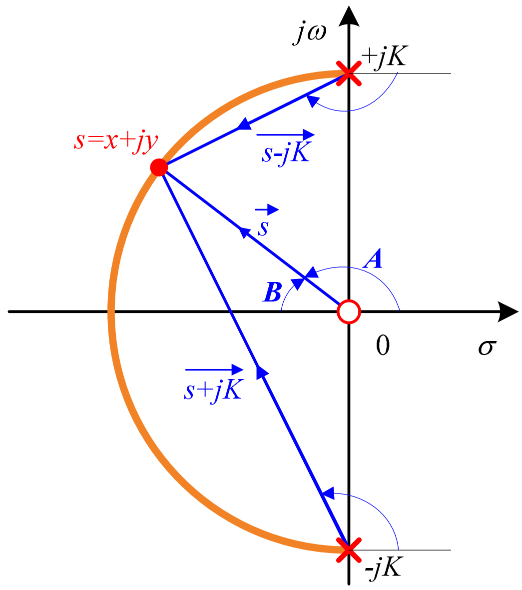
Hence, phase criterion is satisfied.

< Magnitude Criterion >

$a=?$ at $s=-K$

sol) $$|aG(s)|=\left|\frac{as}{s^2+K}\right|=\frac{a|s|}{|s+j\sqrt{K}||s-j\sqrt{K}|}=1 \quad \Rightarrow \quad a=\frac{\sqrt{2}K\times\sqrt{2}K}{K}=2K$$



## 7.3 The Root locus procedure ~ 7 steps

<step 1> Write the char. eq. as follows ;

$1+Kp(s)=0$  where $K$ is a variable parameter.

$$\Rightarrow \quad 1+K\frac{\prod_{i=1}^{n_z}(s+z_i)}{\prod_{j=1}^{n_p}(s+p_j)}=0 \quad \text{where } z_i \text{ : 영점, } p_j \text{ : 극점}$$

⇒ pole−zero map

When $K=0$, the roots of char. eq. are the poles of $p(s)$. ⇒ 근궤적의 시작점

When $K=\infty$, the roots of char. eq. are the zeros of $p(s)$. ⇒ 근궤적의 끝나는 점

proof)

$$\prod_{j=1}^{n_p}(s+p_j)+K\prod_{i=1}^{n_z}(s+z_i)=0$$

if $K=0$, $\Rightarrow \prod_{j=1}^{n_p}(s+p_j)=0$ : $p(s)$의 극점

if $K=\infty$, $\Rightarrow K\prod_{i=1}^{n_z}(s+z_i)=0$ : $p(s)$의 영점

<step 2> Locate the segments of real axis that are root loci.

⇒ The root locus on the real axis always lies in a section to the left of an odd number of poles and zeros of $p(s)$.

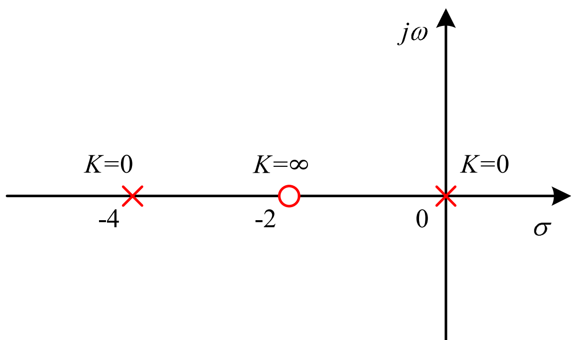
∵) phase criterion

*Example 7.1)* 2nd order system

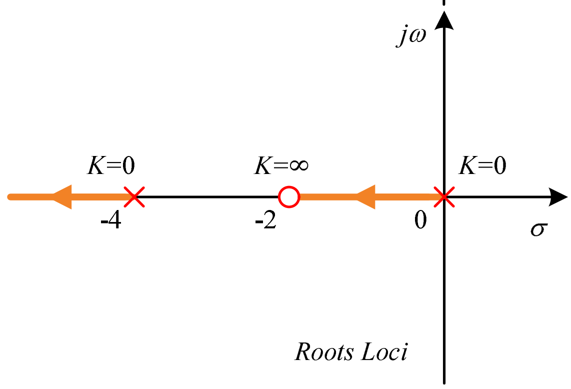
char. eq. $1+GH(s)=1+K\dfrac{\frac{1}{2}s+1}{\frac{1}{4}s^2+s}=0$

1) $1+K\dfrac{2s+4}{s^2+4s}=0 \quad \Rightarrow \quad 1+K\dfrac{2(s+2)}{s(s+4)}=0$

2) root locus on the real axis



*Pole-zero map*



*Roots Loci*

※ check angle criterion

$\measuredangle(s+2)-[\measuredangle(s)+\measuredangle(s+4)] = \pm 180^\circ$

※ at $s=-1$, $K=?$

$$\frac{2K|s+2|}{|s||s+4|}=1 \quad \Rightarrow \quad K=\frac{1\times 3}{2\times 1}=\frac{3}{2}$$

<step 3> Asymptotes of root locus

$\Rightarrow$ $K=\infty$일 때 근궤적의 수렴선

The loci proceed to the zeros at $s=\infty$ along the asymptotes centered at $\sigma_A$ and with the angle $\phi_A$.

$$\sigma_A = \frac{\sum \text{poles of } p(s) - \sum \text{zeros of } p(s)}{n_p - n_z}$$

$$\phi_A = \frac{(2q+1)\pi}{n_p - n_z}, \qquad q = 1, 2, \cdots (n_p - n_z - 1)$$

*Example*) In the previous example 7.1)

$$\sigma_A = \frac{(0-4)-(-2)}{2-1} = -2 \qquad \text{and} \qquad \phi_A = \frac{(2q+1)\pi}{2-1}, \quad q=0$$
$$= \pi$$

*Example*) Plot root locus.

char. eq. $s^2 + 2s + K = 0$

sol)

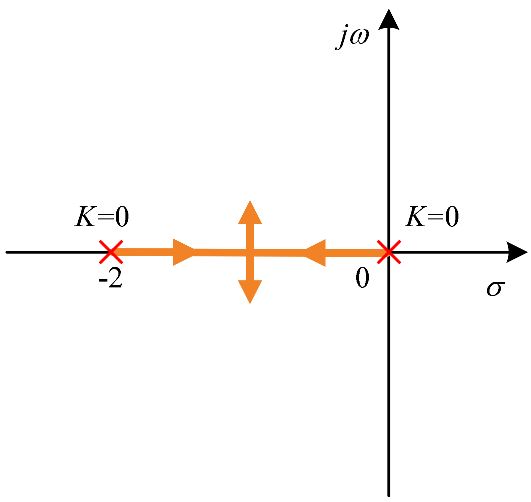
1) $1 + K\dfrac{1}{s^2+2s} = 0 \quad \Rightarrow \quad 1 + K\dfrac{1}{s(s+2)} = 0$

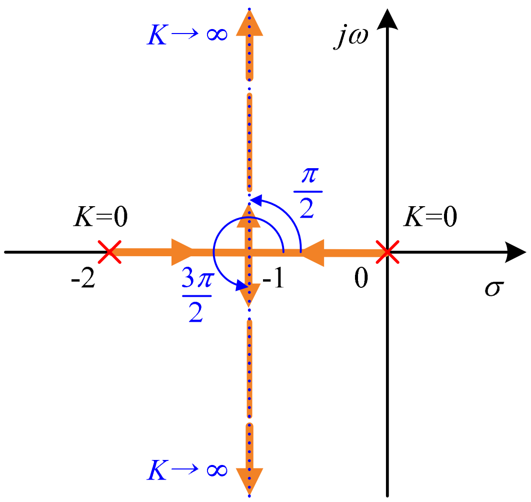
$\Rightarrow$ pole−zero map

2) root locus on the real axis

3) asymptotes of root locus

$$\sigma_A = \frac{(0-2)-(0)}{2-0} = -1 \qquad \text{and} \qquad \phi_A = \frac{(2q+1)\pi}{2-0}, \quad q=0,1$$
$$= \frac{\pi}{2}, \ \frac{3\pi}{2}$$



*Example 7.2* ) $4^{\text{th}}$ order system

$$1+K\frac{s+1}{s(s+2)(s+4)^2}=0$$

1) pole−zero map

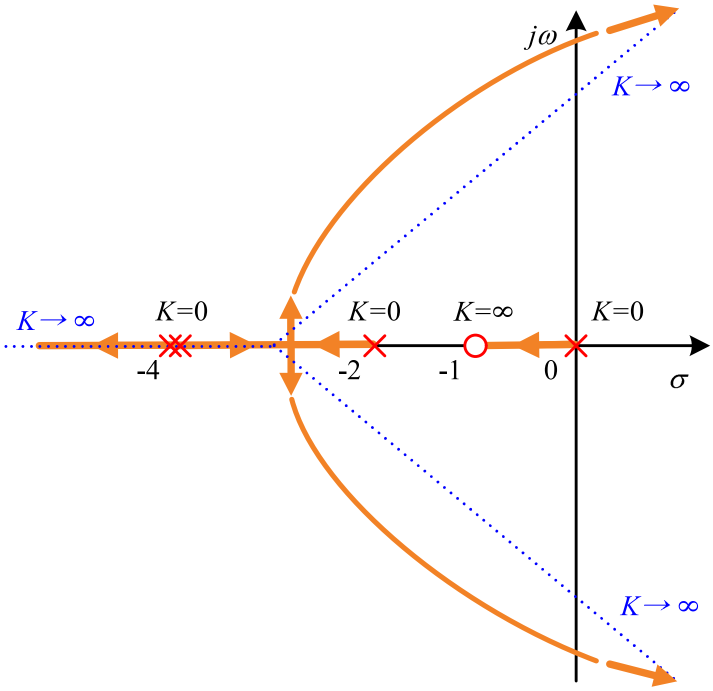
2) root locus on the real axis

3) asymptotes

$$\sigma_A=\frac{(0-2-4-4)-(-1)}{4-1}=-3$$

$$\phi_A=\frac{(2q+1)\pi}{4-1},\qquad q=0,1,2$$

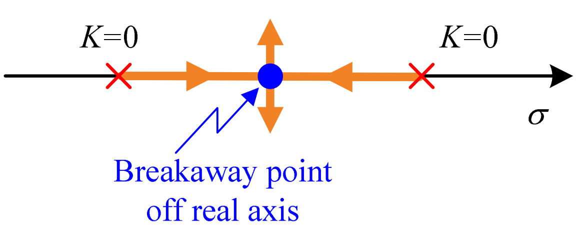
$$=\frac{\pi}{3},\ \frac{3\pi}{3},\ \frac{5\pi}{3}$$



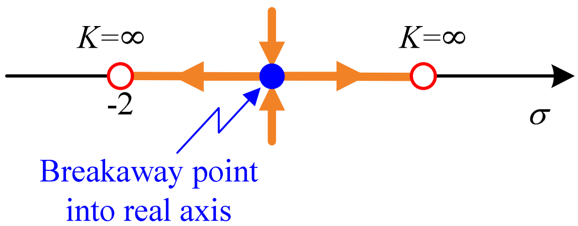

<step 4> Intersection with imag. axis

⇒ Routh−Hurwitz criterion

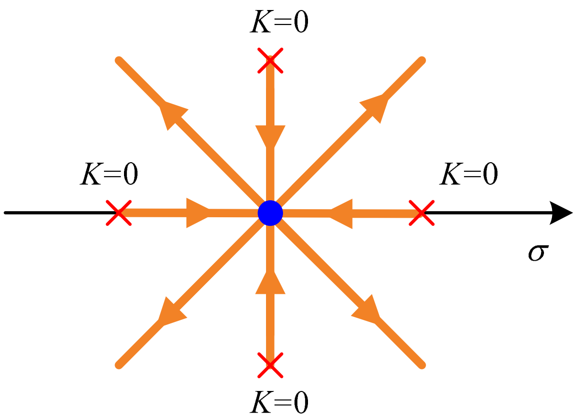
<step 5> Breakaway point off/into real axis

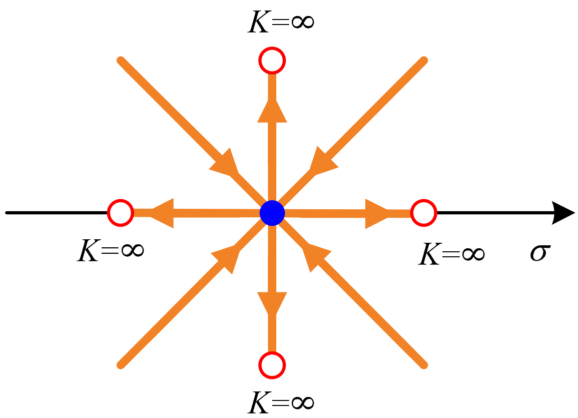



To find real values $s$ that maximizes $K$ between 2 poles




To find real values $s$ that minimizes $K$ between 2 zeros




$$1+Kp(s)=0\quad\Rightarrow\quad K=\frac{-1}{p(s)}\equiv A(s)$$

Analytically $\dfrac{dA(s)}{ds}=0$

*Example)* $1 + K\dfrac{1}{(s+2)(s+4)} = 0$

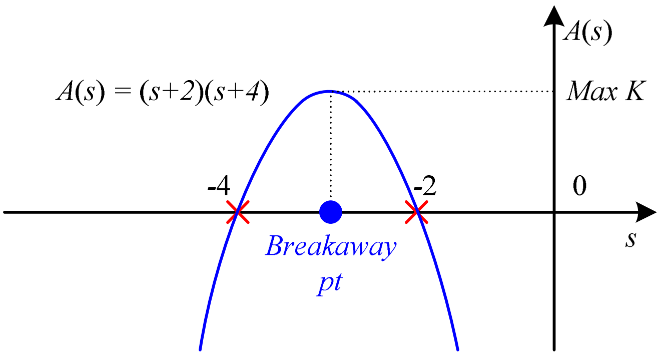
sol)

$K = -(s+2)(s+4) \equiv A(s)$

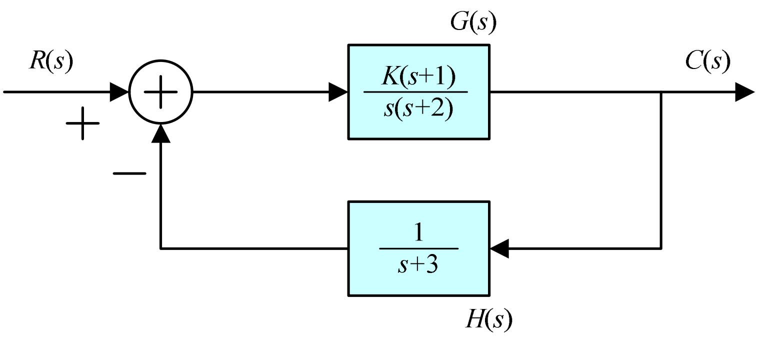
$\dfrac{dA(s)}{ds} = (-1)(s+4) + [-(s+2)](1) = 0$

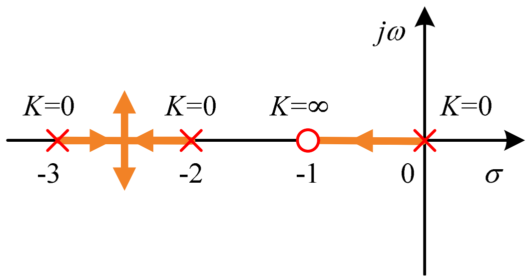
$2s + 6 = 0 \quad \Rightarrow \quad s = -3 \quad \Rightarrow \quad K = -(-1)(1) = 1$

Another method) Graphical method




*Example 7.3)* 3rd order system




sol)

1) char. eq.

$$1 + GH(s) = 1 + K\frac{s+1}{s(s+2)(s+3)} = 0$$

$\Rightarrow$ pole−zero map

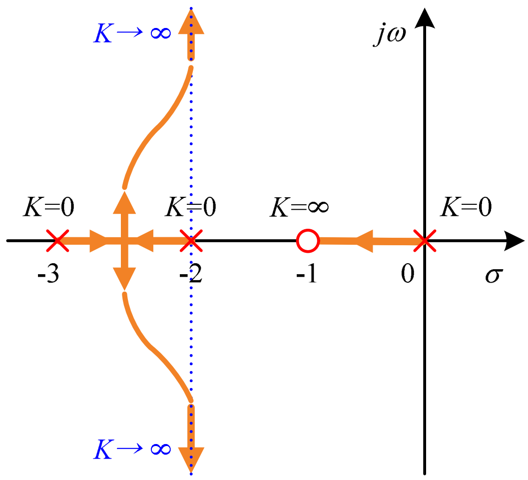
2) root locus on the real axis

3) asymptotes

$$\sigma_A = \frac{(0-2-3)-(-1)}{3-1} = -2$$

$$\phi_A = \frac{(2q+1)\pi}{3-1}, \qquad q = 0,1$$

$$= \pi/2, \;\; 3\pi/2$$



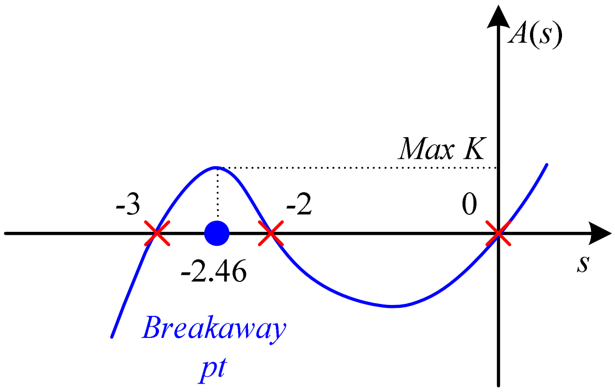
4) breakaway point, $[-3,\ -2]$

$$K = \frac{-s(s+2)(s+3)}{s+1} = A(s)$$

$$\frac{dA(s)}{ds} = \frac{\text{자}'\text{모} - \text{자모}'}{\text{모}^2} = 0$$

$$\Rightarrow \quad s^3 + 4s^2 + 5s + 3 = 0$$

$$\Rightarrow \quad s_{1,2,3} = -2.46,\ \times,\ \times$$



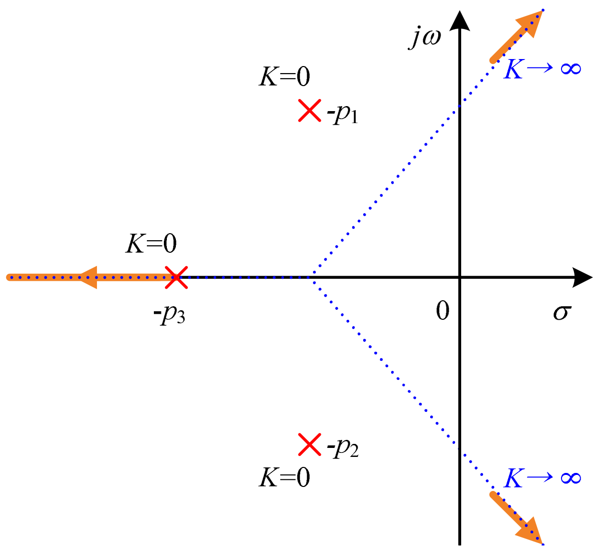

Another solution)

| $s$ | $-2$ | $\cdots$ | $-2.4$ | $\cdots$ | $-2.45$ | $\cdots$ | $-2.5$ | $\cdots$ | $-2.6$ | $\cdots$ | $-3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $K = A(s)$ | 0 | | 0.412 | | 0.42 | | 0.417 | | 0.39 | | 0 |

<step 6> Departure angle from complex poles

Arrival angle at complex zeros

$\Rightarrow$ using phase criterion




$$q(s) = 1 + K\frac{1}{(s+p_3)(s^2 + 2\zeta w_n s + w_n^2)} = 0$$

where $p(s) = \dfrac{1}{(s+p_3)(s+p_1)(s+p_2)}$

$p_2 = p_1^*$ : complex conjugate poles

for small incremental distance $s$ from $p_1$

$\theta_1 = \measuredangle\left(\overrightarrow{s+p_1}\right)$ : departure angle

phase criterion

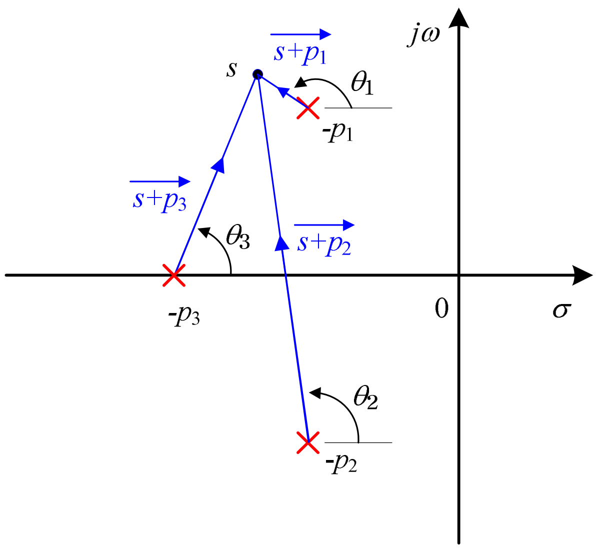
$\measuredangle\{p(s)\} = 180^\circ$

$-\left[\measuredangle\left(\overrightarrow{s+p_3}\right) + \measuredangle\left(\overrightarrow{s+p_1}\right) + \measuredangle\left(\overrightarrow{s+p_2}\right)\right] = 180^\circ$

$\theta_3 + \theta_1 + \theta_2 = 180^\circ \quad \leftarrow \quad \theta_2 \approx 90^\circ$

Hence, departure angle is

$\theta_1 = 90^\circ - \theta_3$



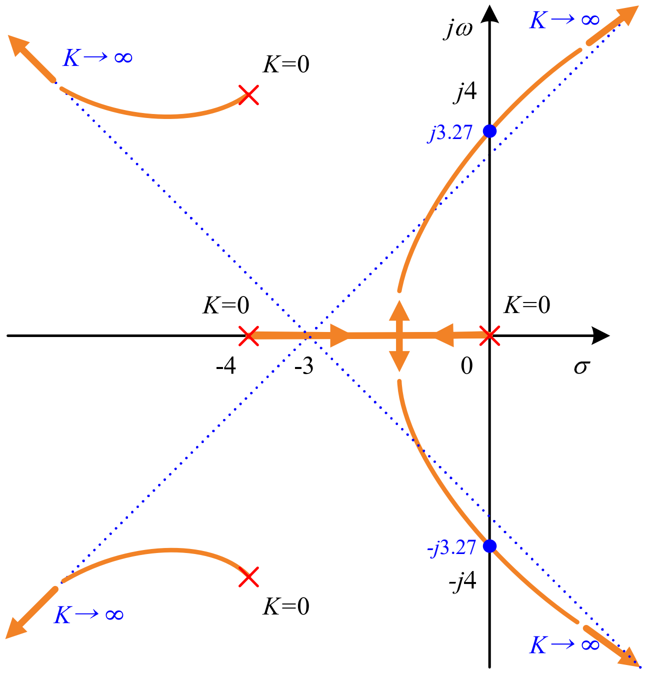
<step 7> (final step) Complete the locus

Determine $K$ at a specific root location.

$\Rightarrow$ using magnitude criterion

$|Kp(s)|=1$

*Example 7.4* ) $4^{th}$ order system




$$1+K\frac{1}{s^4+12s^3+64s^2+128s}=0$$

1) pole-zero map

$$1+K\frac{1}{s(s+4)(s+4+j4)(s+4-j4)}=0$$

2) root locus on the real axis

3) asymptotes

$$\sigma_A=\frac{(0-4-4-j4-4+j4)-(0)}{4-0}=-3$$

$$\phi_A=\frac{(2q+1)\pi}{4-0}, \qquad q=0,1,2,3$$

$$=\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$$

4) intersection with imag. axis

$s^4+12s^3+64s^2+128s+K=0$

| | | | |
|---|---|---|---|
| $s^4$ | 1 | 64 | $K$ |
| $s^3$ | 12 | 128 | 0 |
| $s^2$ | 53.33 | $K$ | |
| $s^1$ | $\frac{53.33\times 128-12K}{53.33}$ | 0 | |
| $s^0$ | $K$ | | |

보조방정식 $53.33s^2+K=0$

if $K=568.89 \Rightarrow$ all zero

To be stable, $0\leqq K\leqq 568.89$

① $\frac{53.33\times 128-12K}{53.33}\geqq 0 \qquad \Rightarrow \quad K\leqq 568.89$

② $K\geqq 0$

Hence, the locus intersects with imag. axis at $K=568.89$.

auxiliary eq.

$53.33s^2+568.89=0 \qquad \Rightarrow \quad s_{1,2}=\pm j3.27$

5) breakaway point, [−4, 0]

$K=-(s^4+12s^3+64s^2+128s)=A(s)$

$\frac{dA(s)}{ds}=-(4s^3+36s^2+128s+128)=-4(s^3+9s^2+32s+32)=0$

$\Rightarrow \quad s_{1,2,3}=-1.5, \ \times, \ \times$

Another solution)

| $s$ | −4 | −3 | −2 | ⋯ | −1.5 | ⋯ | −1 | 0 |
|---|---|---|---|---|---|---|---|---|
| $K=A(s)$ | 0 | 51 | 80 | | 83.44 | | 75 | 0 |

6) departure angle from complex poles

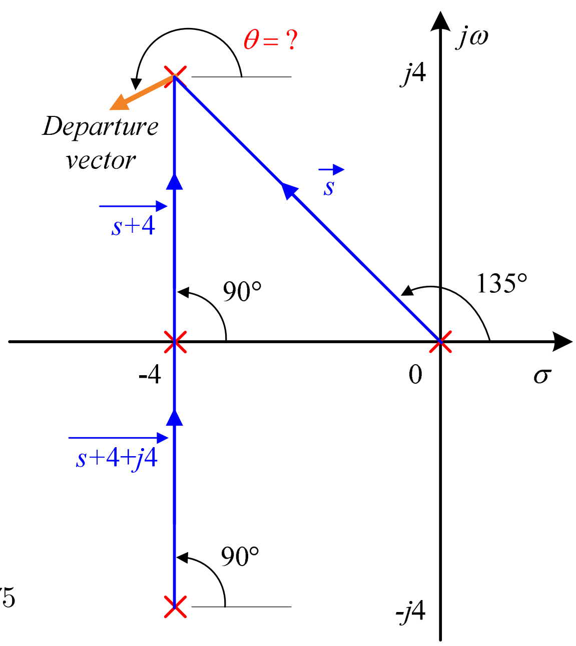
phase criterion

$\measuredangle\{p(s)\}=180^\circ$

$\measuredangle(\vec{s})+\measuredangle(\overrightarrow{s+4})+\measuredangle(\overrightarrow{s+4+j4})+\measuredangle(\overrightarrow{s+4-j4})=180^\circ$

$135^\circ+90^\circ+90^\circ+\theta=180^\circ$

Hence, $\theta=-135^\circ$ or $225^\circ$




7) complete the locus

※ Find $K$ at $s=-1$

sol)

Mag. criterion

$|Kp(s)|=1$

$\frac{K}{|s|\ |s+4|\ |s+4+j4|\ |s+4-j4)|}=1$

$K=|s|\ |s+4|\ |s+4+j4|\ |s+4-j4)|$

at $s=-1$, $\quad K=1\times3\times\sqrt{4^2+3^2}\times\sqrt{4^2+3^2}=75$

※ Find $K$ at $s=-1.5$

sol)

$K=1.5\times2.5\times\sqrt{4^2+2.5^2}\times\sqrt{4^2+2.5^2}=83.44$

another solution)

$K=-(s^4+12s^3+64s^2+128s)\|_{s=-1.5}$

$=83.44$

※ Dominant roots : 우세근/지배근

: roots near the origin

*Example*) Return to previous example.

At $K = 126$

char. eq.

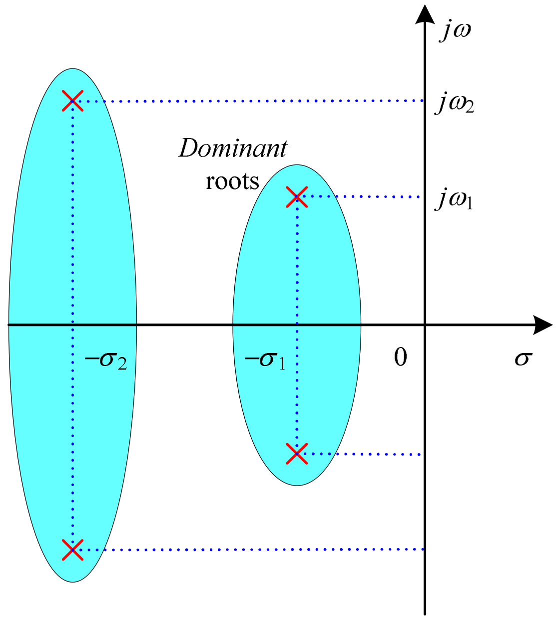
$$(s+\sigma_1+jw_1)(s+\sigma_1-jw_1) \cdot (s+\sigma_2+jw_2)(s+\sigma_2-jw_2)=0$$

output

$$c(t) = A_1 e^{-\sigma_1 t}\sin(w_1 t+\theta_1) + A_2 e^{-\sigma_2 t}\sin(w_2 t+\theta_2)$$

※ 실수부분의 크기가 클수록 시상수가 짧다.

⇒ 빨리 정상상태에 도달(사라짐)




※ 원점에 가까운 극점이 과도응답 특성을 지배함.

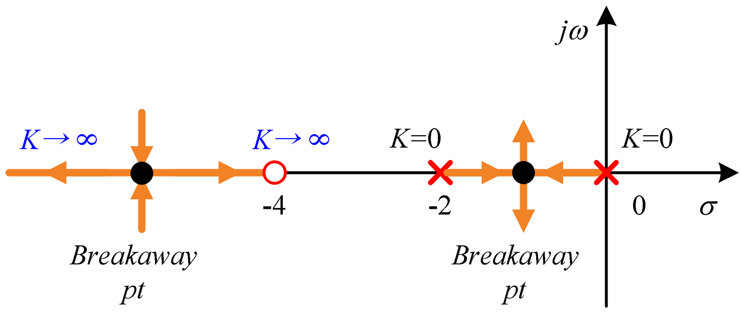
*Example*) Find root locus.

$q(s) = s(s+2) + K(s+4) = 0$

sol)

1) pole−zero map of $p(s)$

$$1 + K\frac{s+4}{s(s+2)} = 0$$




2) root locus on the real axis

3) asymptotes

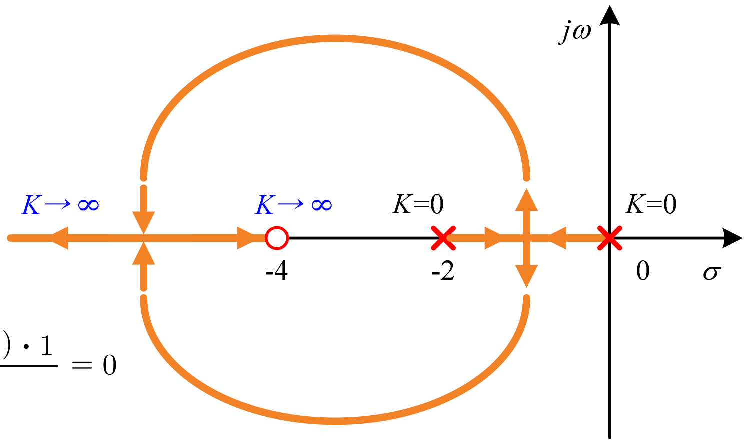
$$\sigma_A = \frac{(0-2)-(-4)}{2-1} = 2$$

$$\phi_A = \frac{(2q+1)\pi}{2-1}, \qquad q=0$$

$$= \pi$$

4) 2 breakaway point

$\sim\ [-\infty, 0]$ and $[-2,\ 0]$




$$K = -\frac{s(s+2)}{s+4} \equiv A(s)$$

$$\frac{dA(s)}{ds} = -\frac{(2s+2)\cdot(s+4) - s(s+2)\cdot 1}{(s+4)^2} = 0$$

$$\Rightarrow \quad s^2 + 8s + 8 = 0$$

$$\Rightarrow \quad s_{1,2} = -4 \pm 2\sqrt{2}$$
$$= -6.828, \quad -1.172$$

※ intersection with imag. axis

$s^2+(K+2)s+4K=0$

| | | |
|---|---|---|
| $s^2$ | $1$ | $4K$ |
| $s^1$ | $K+2$ | |
| $s^0$ | $4K$ | |

To be stable, $K \geqq 0$

① $K+2 \geqq 0 \quad \Rightarrow \quad K \geqq -2$

② $K \geqq 0$

※ Find $K$ at $s=-1$.

sol)

$$K=-\left.\frac{s(s+2)}{s+4}\right|_{s=-1}=-\frac{-1\cdot 1}{3}=\frac{1}{3}$$

※ Find $K$ to be settling time $T_s=\dfrac{4}{\zeta w_n}=2\,\text{sec.}$.

sol)

$\zeta w_n=2 \quad \Rightarrow \quad -\zeta w_n=-2 \quad \Rightarrow \quad s_{1,2}=-2\pm jx$ (특성방정식의 실수부분 = 2)

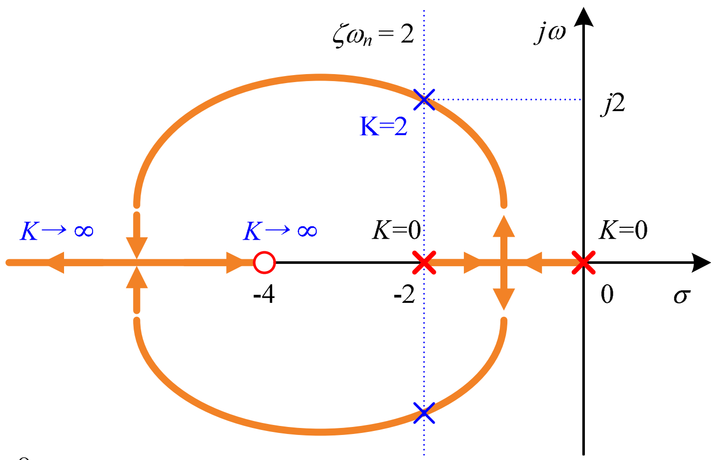
$(s+2+jx)(s+2-jx)=(s+2)^2-(jx)^2=0$

$s^2+4s+4+x^2=0 \quad$ cf) $s^2+(K+2)s+4K=0$

Hence,

① $K+2=4 \quad \Rightarrow \quad K=2$

② $4K=4+x^2 \quad \Rightarrow \quad x=2$




※ Find $K$ to be $\zeta=\dfrac{1}{\sqrt{2}}=0.707$.

sol)

$$\cos(\theta)=\frac{\zeta w_n}{w_n}=\zeta \ \leftarrow \ \frac{1}{\sqrt{2}}$$

$\theta=45^\circ \quad \Rightarrow \quad s_{1,2}=-x\pm jx$

$(s+x+jx)(s+x-jx)=(s+x)^2-(jx)^2=0$

$s^2+2xs+2x^2=0 \quad$ cf) $s^2+(K+2)s+4K=0$

① $K+2=2x \quad \Rightarrow \quad K=2x-2$

② $4K=2x^2 \quad \Rightarrow \quad 4(2x-2)=2x^2 \quad \Rightarrow \quad x^2-4x+4=0 \quad \Rightarrow \quad x_{1,2}=2$

$K=2(2)-2=2$

*Example*) Find root locus.

$$s(s+5)(s+6)(s^2+2s+2)+K(s+3)=0$$

sol)

1) pole−zero map of $p(s)$

$$1+K\frac{s+3}{s(s+5)(s+6)(s^2+2s+2)}=0$$

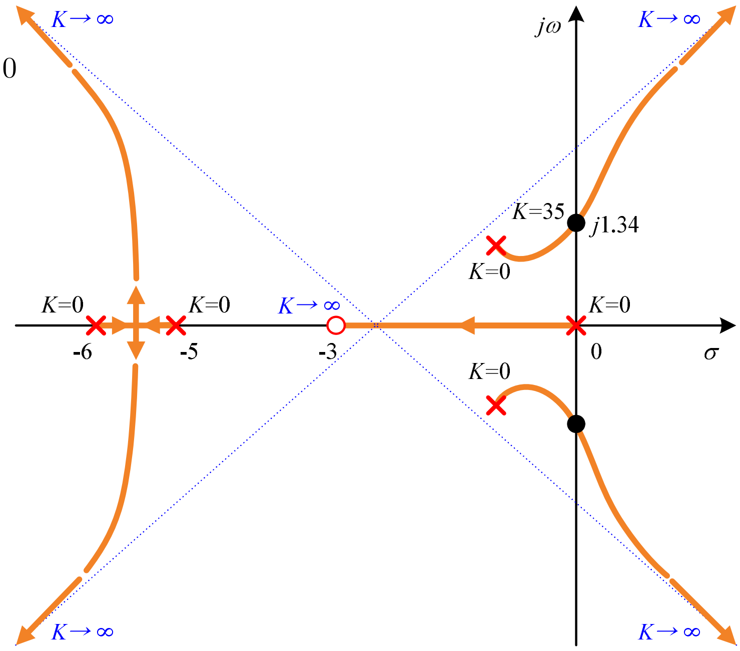
2) root locus on the real axis

3) asymptotes

$$\sigma_A=\frac{(-13)-(-3)}{5-1}=-2.5$$

$$\phi_A=\frac{(2q+1)\pi}{5-1},\qquad q=0,1,2,3$$

$$=\pi/4,\ \ 3\pi/4,\ \ 5\pi/4,\ \ 7\pi/4,$$




※ departure angle from complex poles

phase criterion

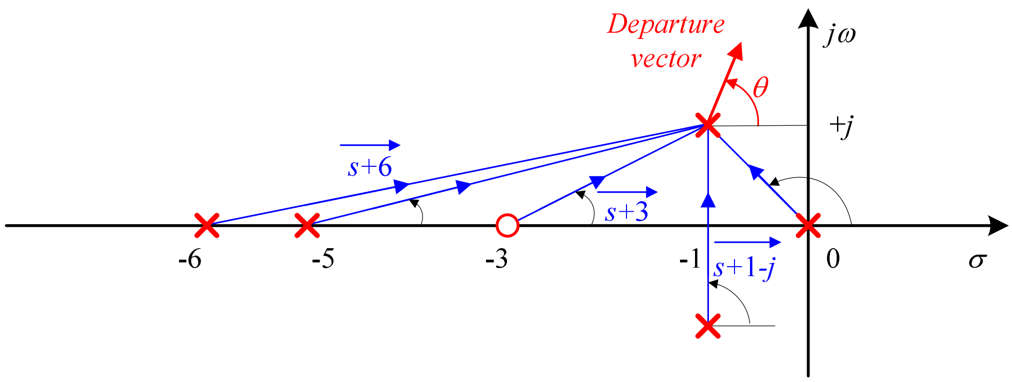
$$\measuredangle\{p(s)\}=180^\circ$$

$$\measuredangle(\overrightarrow{s+3})-[\measuredangle(\vec{s})+\measuredangle(\overrightarrow{s+5})+\measuredangle(\overrightarrow{s+6})+\measuredangle(\overrightarrow{s+1+j})+\measuredangle(\overrightarrow{s+1-j})]=180^\circ$$

where $\measuredangle(\overrightarrow{s+1-j})=\theta=?$

$$\Rightarrow\quad 26.6^\circ-[135^\circ+14^\circ+11.4^\circ+90^\circ+\theta]=180^\circ$$

$$\Rightarrow\quad \theta=-43.8^\circ$$




$$\measuredangle(\overrightarrow{s+3})|_{s=-1+j}=\measuredangle(\overrightarrow{-1+j+3})=\measuredangle(\overrightarrow{2+j})=\tan^{-1}(\frac{1}{2})\approx 26.6^\circ$$

$$\measuredangle(\vec{s})|_{s=-1+j}=\measuredangle(\overrightarrow{-1+j})=\tan^{1}(\frac{1}{-1})=135^\circ$$

$$\measuredangle(\overrightarrow{s+5})|_{s=-1+j}=\measuredangle(\overrightarrow{-1+j+5})=\measuredangle(\overrightarrow{4+j})=\tan^{-1}(\frac{1}{4})\approx 14^\circ$$

$$\measuredangle(\overrightarrow{s+6})|_{s=-1+j}=\measuredangle(\overrightarrow{-1+j+6})=\measuredangle(\overrightarrow{5+j})=\tan^{-1}(\frac{1}{5})\approx 11.4^\circ$$

$$\measuredangle(\overrightarrow{s+1+j})|_{s=-1+j}=\measuredangle(\overrightarrow{-1+j+1+j})=\measuredangle(\overrightarrow{j2})=\tan^{2}(\frac{2}{0})=90^\circ$$

※ intersection with imag. axis

$$s^5+13s^4+54s^3+82s^2+(60+K)s+3K=0$$

| | | | | |
|---|---|---|---|---|
| $s^5$ | $1$ | $54$ | $60+K$ | |
| $s^4$ | $13$ | $82$ | $3K$ | |
| $s^3$ | $47.7$ | $60+0.769K$ | | |
| $s^2$ | $65.6-0.212K$ | $3K$ | | $\Leftarrow$ 보조방정식 |
| $s^1$ | $\dfrac{3940-105K-0.163K^2}{65.6-0.212K}$ | | | $\Rightarrow$ all zero if $K=35$ |
| $s^0$ | $3K$ | | | |

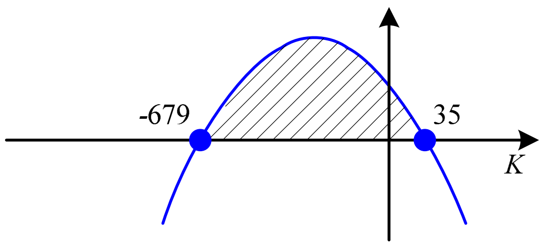
To be stable, $0 \leqq K \leqq 35$

① $65.6-0.212K \geqq 0 \quad \Rightarrow \quad K \leqq 309$

② $3940-105K-0.163K^2 \geqq 0$

$\quad \Rightarrow \quad -679 \leqq K \leqq 35$

③ $K \geqq 0$




Hence, the locus intersects with imag. axis at $K=35$.

auxiliary eq.

$$(65.6-0.212K)s^2+3K=0 \quad \leftarrow \quad K=35$$

$$58.2s^2+105=0$$

$$\Rightarrow \quad s_{1,2}=\pm j1.34$$

※ Find $K$ at $s=-1$

sol)

$$|Kp(s)|=1$$

$$K=\frac{|s||s+5||s+6||s+1+j||s+1-j|}{|s+3|}=\frac{1\cdot 4\cdot 5\cdot 1\cdot 1}{2}=10$$