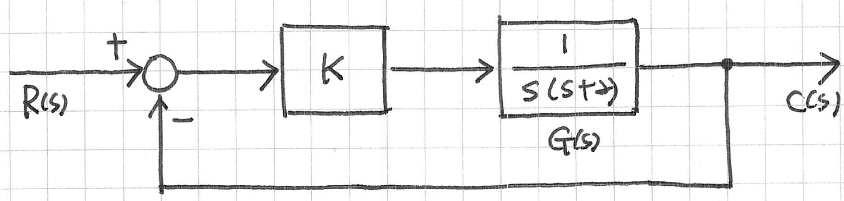
Chap. 7 Root locus Method

The root locus is the trajectory of the roots of the char. eq. in the s-plane as a system parameter varies.

⇒ graphical information

7.2. Root locus concept.

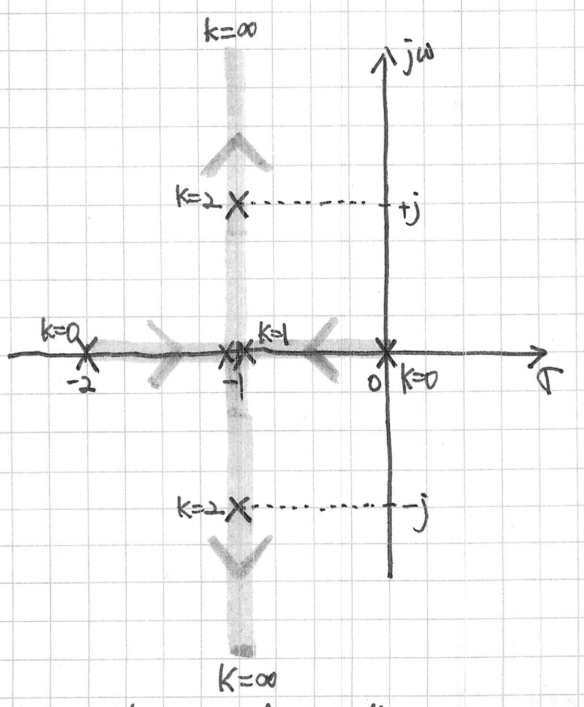
Ex) Unity - f/b control system ~ Fig. 7.2.



$$T(s) = \frac{\sum P_k \Delta_k}{\Delta(s)}, \quad \Delta(s) = 1 + K\,G(s)$$

Char. eq. $\quad q(s) = 1 + K\dfrac{1}{s(s+2)} = 0$

$\underline{s^2 + 2s + K = 0}$ where K is a system parameter.



"Root locus."

$k=0, \quad s_{1,2} = 0, -2$

$k=1, \quad s_{1,2} = -1$

$k=2 \quad s_{1,2} = -1 \pm j$

⋮

char. eq. $\quad 1 + KG(s) = 0$

$\quad\quad\quad\quad 1 + KP(s) = 0$

where K is a variable parameter. $(K \geq 0)$

$\Rightarrow \quad KP(s) = -1 + j0$

$$|KP(s)| \angle KP(s) = 1 \angle 180^\circ$$

$\therefore \begin{cases} |KP(s)| = 1 & \text{(*) magnitude criterion} \\ \angle KP(s) = 180^\circ \pm k\,360^\circ, \quad k = 0, \pm 1, \pm 2, \cdots & \text{(**) angle criterion} \end{cases}$

The root locus is constructed by finding all points in the s-plane that satisfy Eqs. (**), and then the values of K along the locus ~~are~~ is determined by Eq. (*).

Ex) Previous example ~ Fig 7.2.

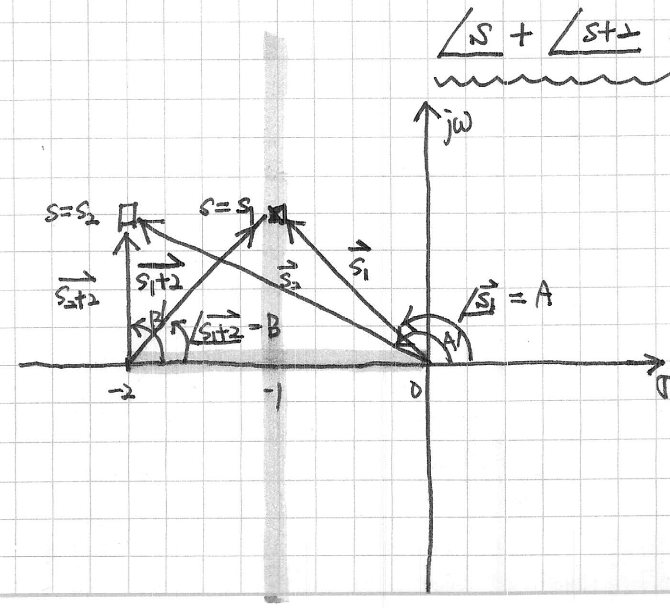
$$g(s) = s^2 + 2s + K = 0$$

$$1 + \frac{K}{s(s+2)} = 0 \quad \left(\because P(s) = \frac{1}{s(s+2)}\right)$$

$$\left|\frac{K}{s(s+2)}\right| = 1 \longrightarrow K = |s| \cdot |s+2|$$

$$\angle \frac{K}{s(s+2)} = \pm 180^\circ, \pm 540^\circ, \cdots \quad \cancel{\angle K} - \angle s - \angle s+2 = \pm 180^\circ, \pm 540^\circ, \cdots$$

$$\angle s + \angle s+2 = \pm 180^\circ, \pm 540^\circ, \cdots$$

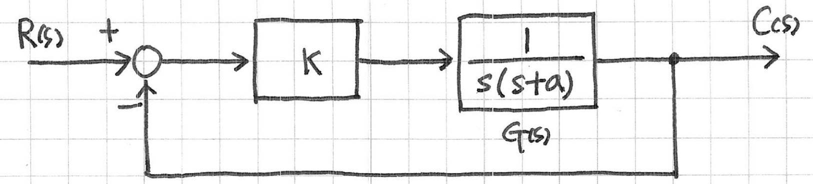


$A + B = 180^\circ$

$A' + B' \neq 180$

$K = 0$

Ex) Single-loop system ~ Fig. 7.5.



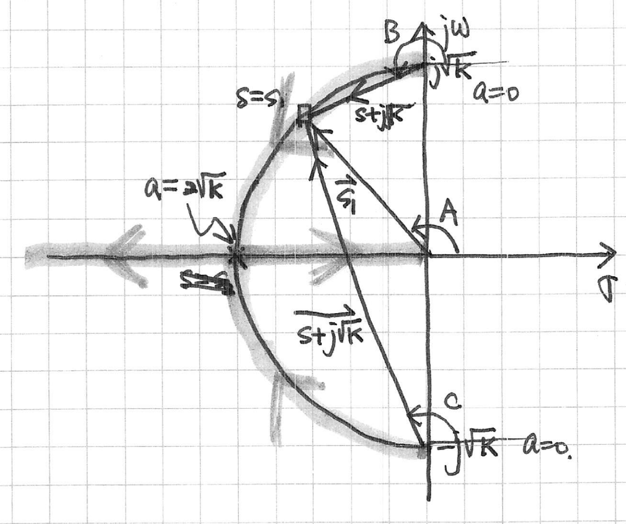
char. eq. $1 + K\,G(s) = 0$

$$s^2 + as + K = 0$$ where a is a parameter. $a \geq 0$

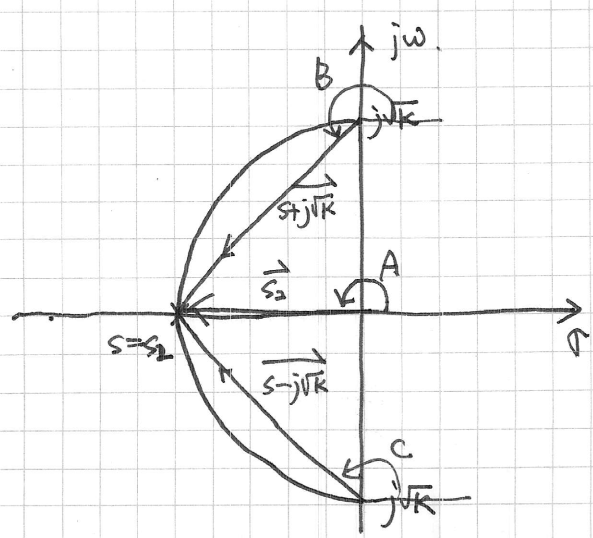
$$1 + a\frac{s}{s^2 + K} = 0$$

$$\left| a\frac{s}{s^2+K} \right| = 1 \longrightarrow \frac{a|s|}{|s^2+K|} = 1$$

$$\angle a\frac{s}{s^2+K} = \pm 180°, \pm 540°, \cdots \rightarrow \overset{A}{\angle s} - \overset{C}{\angle s+j\sqrt{K}} - \overset{B}{\angle s-j\sqrt{K}} = \pm 180, \cdots$$



$$A - (B + C) = 135° - (315°) = -180°$$



$$A - (B+C) = 180° - (225° + 135°)$$
$$= 180° - 360° = -180°$$

7.3 The Root locus procedure ~ 12 steps.

<step 1> Write the char. eq. as follows;

$$1 + KP(s) = 0$$

where K is a variable ~~par~~ parameter.

<step 2> Write $P(s)$ in the form of poles and zeros

$$1 + K \frac{\prod_{i=1}^{n_z} (s+z_i)}{\prod_{j=1}^{n_p} (s+p_j)} = 0 \qquad (7.24)$$

<step 3> Locate the poles and zeros of $P(s)$ on the s-plane.

<step 4> Locate the segments of real axis that are root loci.

(a) when $K=0$, the roots of char. eq. are the poles of $P(s)$.
when $K=\infty$, $\pm\infty$ 〃 〃 〃 zeros 〃 .

참) Eq. (7-25) from Eq(7.24)

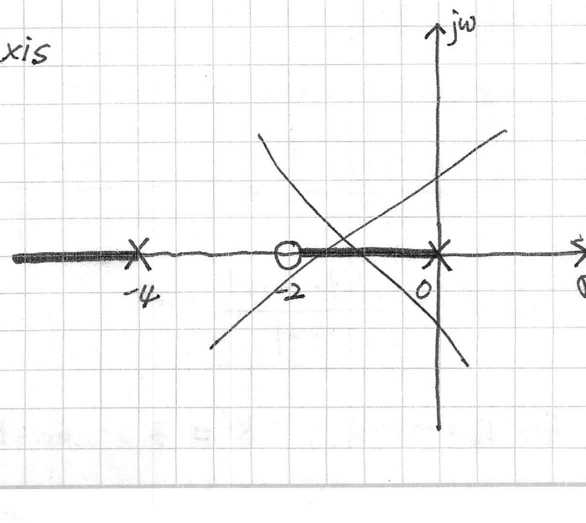
$$\prod_{j=1}^{n_p} (s+p_j) + K \prod_{i=1}^{n_z} (s+z_i) = 0.$$

if $K=0$, $\prod_{i=1}^{n_p} (s+p_j) = 0$.

if $K=\infty$, $K=-\infty$, $K \prod_{i=1}^{n_z} (s+z_i) = 0$.

$\Rightarrow$ The root locus of char. eq. begins at the poles of $P(s)$ $(K=0)$ and ends at the zeros of $P(s)$ $K=\infty$. as K increses from 0 to ∞.

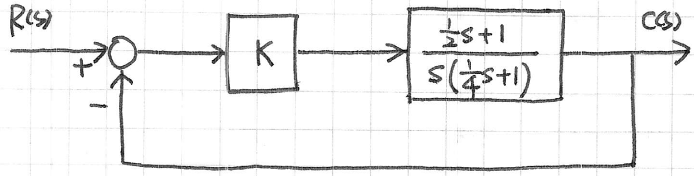
(b) The root locus on the real axis always lies in a sections to the left of an odd number of poles and zeros of $P(s)$.



참) angle criterion

~~Ex) $q(s) = 1 + K \frac{s+2}{s(s+4)} = 0$.~~

Example) 1.1 2nd order system



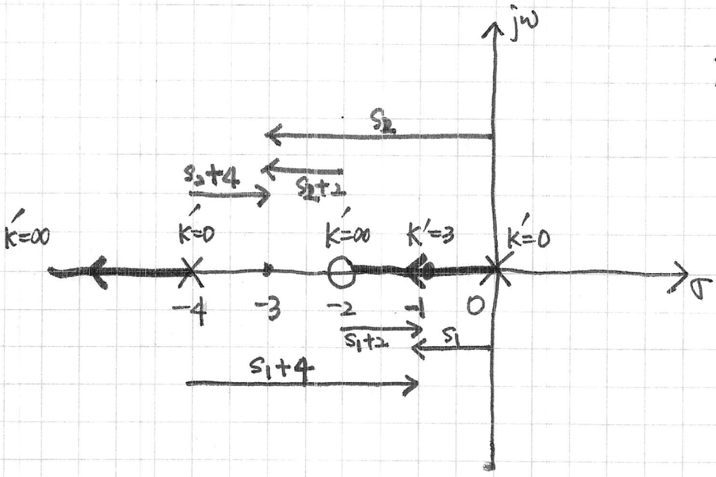
sol) char. eq.

$$1 + K\frac{\frac{1}{2}s+1}{s(\frac{1}{4}s+1)} = 0 \qquad \therefore\ 1 + 2K\frac{s+2}{s(s+4)} = 0$$

K' : parameter. <step 1>.

$p(s)$. < 〃 2>

$$1 + K'p(s) = 0.$$



For $s_2 = -3$.

$$\angle s_2+2 - \angle s_2 - \angle s_2+4$$
$$= 180° - 180° - 0 = 0°$$
$$\neq \pm 180°,$$

$\therefore$ Angle criterion is not satisfied.

one zeros of $p(s)$ lies at ∞.

At $s_1 = -1$ (on the real axis).

(1). angle criterion.

$$\angle K'p(s) = \angle\frac{K'(s_1+2)}{s_1(s_1+4)} = \cancel{\angle K'}^{0} + \angle s_1+2 - \angle s_1 - \angle s_1+4 = \pm 180°, \pm 540°, \cdots$$

$$\therefore \underset{0°}{\angle s_1+2} - \underset{180°}{\angle s_1} - \underset{0°}{\angle s_1+4} = \pm 180°, \pm 540°, \cdots$$
$$= -180°$$

The angle criterion is satisfied for $s < -4$ and $-2 < s < 0$.

(2) Magnitude criterion

$$\left|\frac{K'(s+2)}{s(s+4)}\right| = K'\frac{|s+2|}{|s|\cdot|s+4|} = 1$$

$$\therefore K' = \frac{|s|\cdot|s+4|}{|s+2|}$$

For $s_0 = -1 = s_1$, $K' = 3$ $\quad\therefore K = K'/2 = \frac{3}{2}$ //

<step 5> Determine the # of separate loci.

$$\# \text{ of branch} = \max(n_p, n_z)$$

<step 6> The root loci must be symmetrical w.r.t the real axis.

∵) pairs of complex conjugate roots.

<step 7> Asymptotes of the root loci

(behavior of root loci at $s=\infty$)

∘ The loci proceed to the zeros at $s=\infty$ along the asymptotes centered at σ_A and with angles ϕ_A.

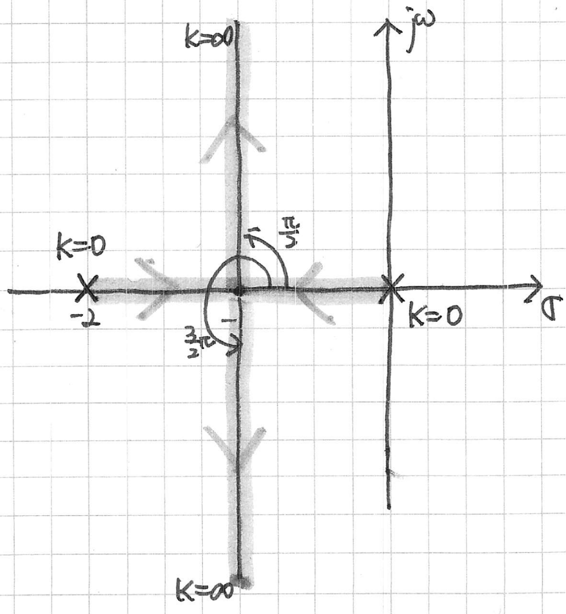
$$\sigma_A = \frac{\sum \text{poles of } p(s) - \sum \text{zeros of } p(s)}{n_p - n_z}$$

$$\phi_A = \frac{(2q+1)\pi}{n_p - n_z}, \quad q = 0, 1, 2, \cdots, (n_p - n_z - 1)$$

$$= \frac{2q}{n_p - n_z}\pi, \quad q = 0, 1, 2, \cdots, (n_p - n_z - 1) \text{ for complementary loci.}$$

Ex) Fig 7.2. $\quad q(s) = s^2 + 2s + K = 0$

$$1 + K\frac{1}{s(s+2)} = 0$$



$$\sigma_A = \frac{-2-0}{2-0} = -1$$

$$\phi_A = \frac{(2q+1)\pi}{2}, \quad q = 0, 1$$

$$= \frac{\pi}{2}, \frac{3}{2}\pi,$$

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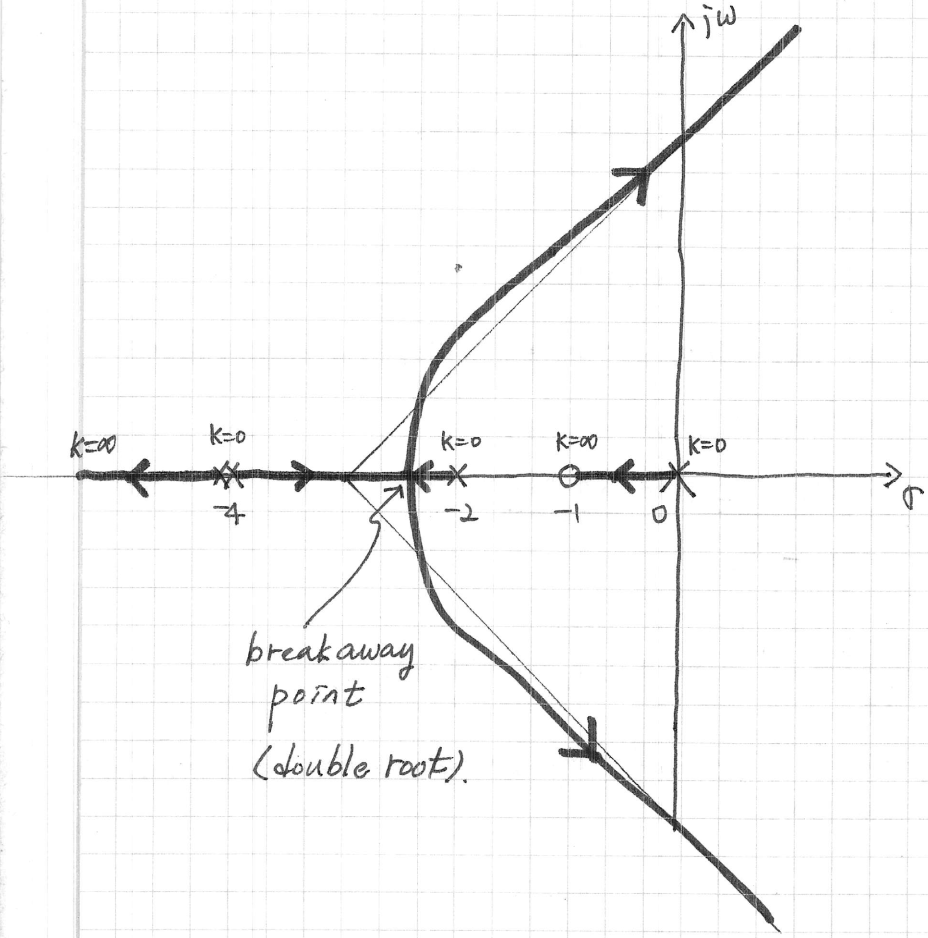
Ex 1.2) 4th order system

$$q(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2} = 0$$



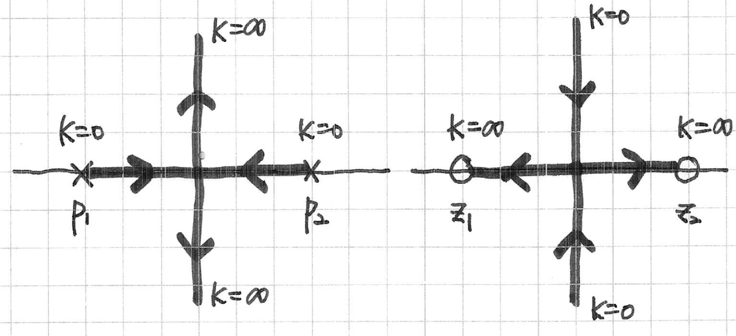
$$\sigma_A = \frac{-10-(-1)}{4-1} = -3$$

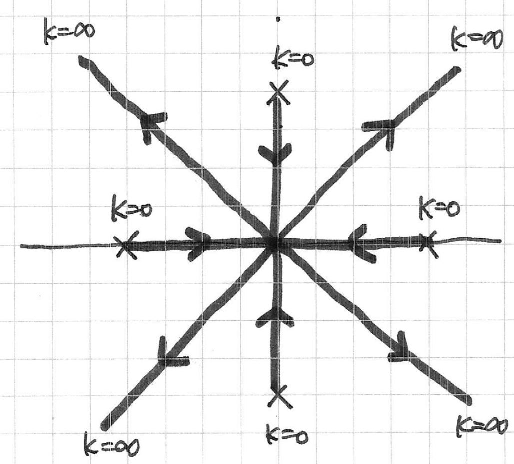
$$\phi_A = \frac{(2q+1)\pi}{4-1}, \quad q=0,1,2$$

$$= \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}$$

< step 8 > Intersection of the loci with the imag. axis
~ Routh-Hurwitz criterion

< step 9 > Breakaway pt





double root

4th order roots

$1 + K P(s) = 0$

$K = \dfrac{-1}{P(s)} \triangleq A(s)$

Find a real value s between $p_1 \sim p_2$ that maximizes $A(s)$. ~ graphically

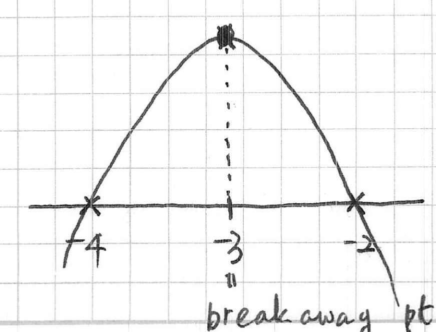
or. $\dfrac{dA(s)}{ds} = 0$ ~ analytically.

Ex) Unity f/b system

$$g(s) = 1 + \frac{K}{(s+2)(s+4)} = 0.$$

$$K = -(s+2)(s+4) \triangleq A(s)$$

① graphical method



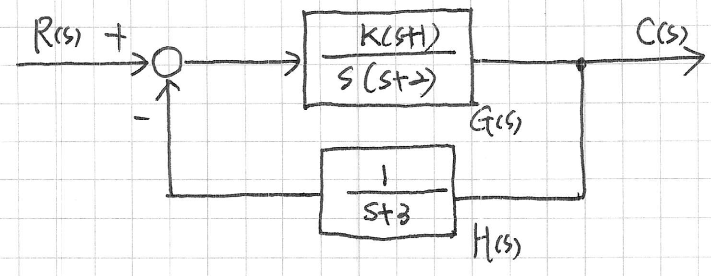
② analytical method

$$\frac{dA(s)}{ds} = -(s+4) - (s+2) = 0$$

$$2s + 6 = 0$$

$\therefore$ $\underset{\sim\sim\sim}{s = -3}$ // $\therefore K = 1.$

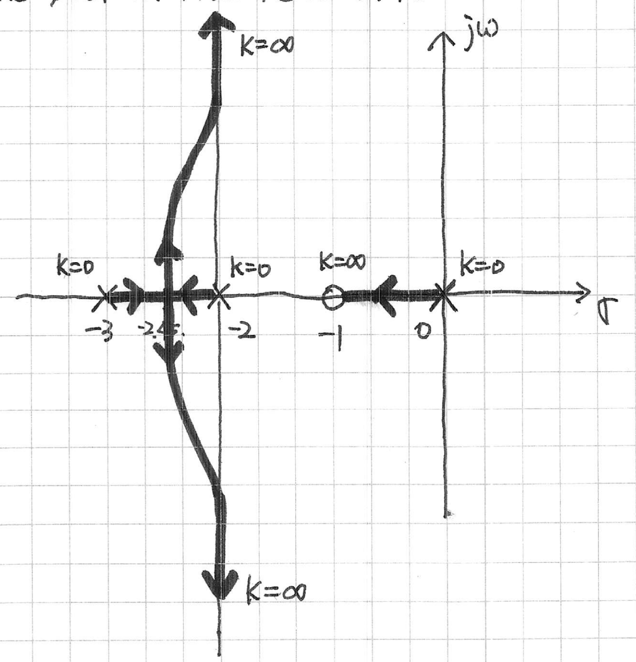
Ex7.3) 3rd order system ~ Fig 7.10



sol) char. eq.

① $q(s) = 1 + G(s)H(s) = 1 + \dfrac{K(s+1)}{s(s+2)(s+3)} = 0$, $1 + K p(s) = 0$.

④ the loci on the real axis



⑦ asymptotes.

$$\sigma_A = \frac{-5-(-1)}{3-1} = -2$$

$$\phi_A = \frac{(2q+1)\pi}{3-1}, \quad q=0,1$$

$$= \frac{\pi}{2}, \ \frac{3}{2}\pi$$

⑨ breakaway pt (if any). : $-3 \sim -2$,

$$K = \frac{-s(s+2)(s+3)}{s+1} = p(s).$$

$$\frac{dp(s)}{ds} = 0$$

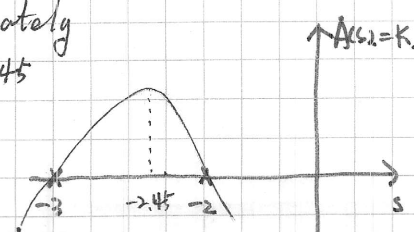
p(s)	0	0.412	0.42	0.417	0.39	0
s	-2	-2.4	-2.45	-2.5	-2.6	-3.

approximately $s = -2.45$

$\Rightarrow s^3 + 4s^2 + 5s + 3 = 0$

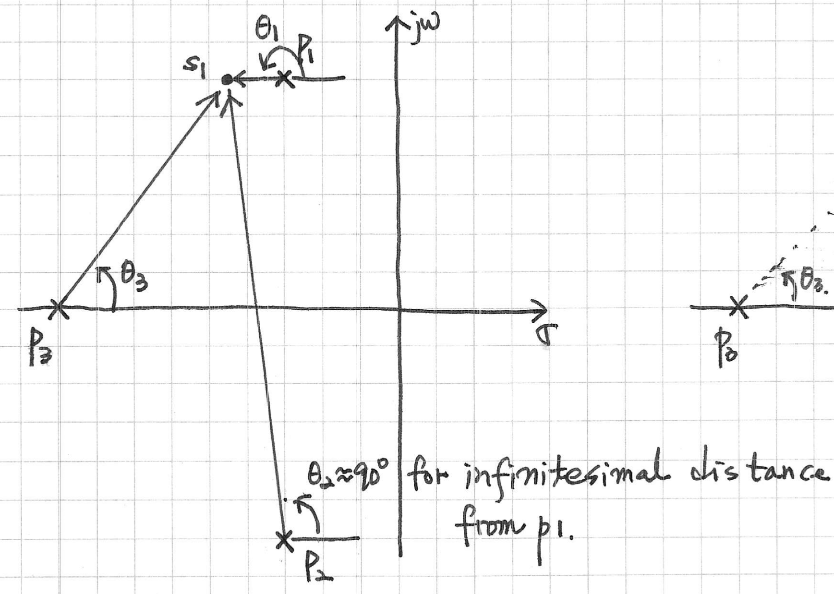
$s_{1,2,3} = -2.45, \ x, \ x$

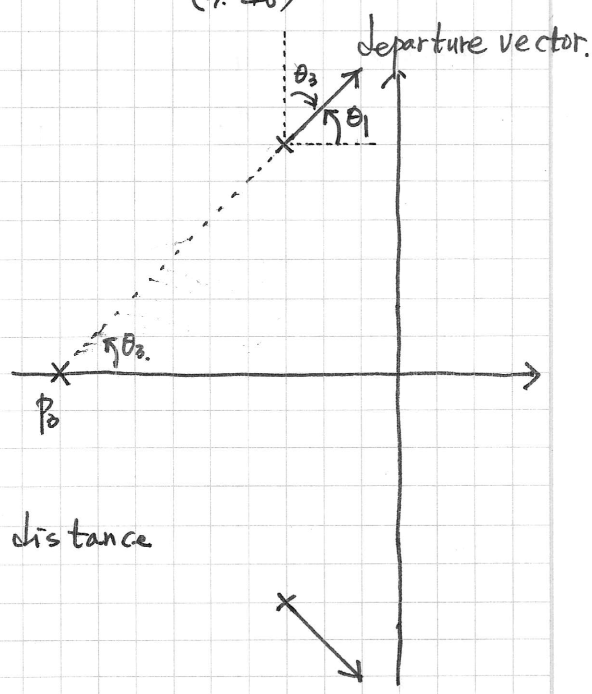
trivial solution.



<step 10.> $\left(\begin{array}{l}\text{departure angle from complex poles} \\ \text{arrival angle at " zeros}\end{array}\right)$ using phase criterion

Ex) $g(s) = 1 + \dfrac{K}{(s+p_3)(s^2+2\zeta\omega_n s+\omega_n^2)}$ (7.48)





phase criterion $\angle p(s) = 180°$

$$\theta_1 + \theta_2 + \theta_3 = 180°$$

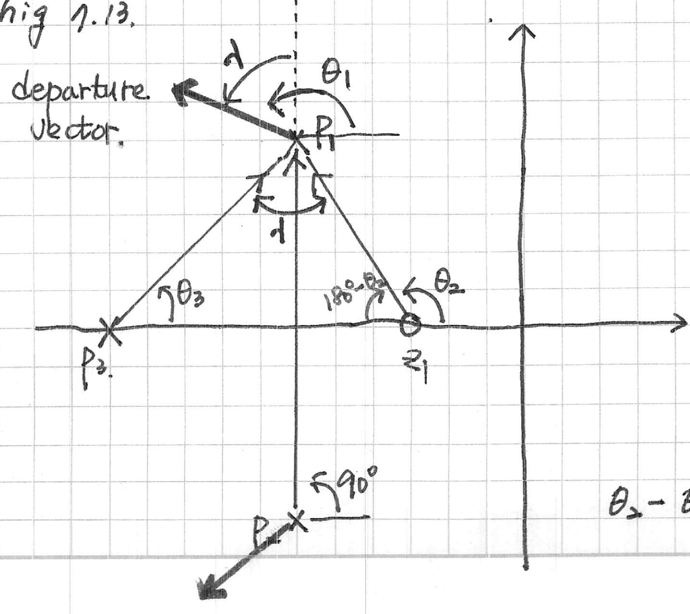
$$\theta_1 + \theta_3 = 90°$$

$$\therefore \underset{\sim\sim\sim}{\theta_1 = 90° - \theta_3}$$ //

$\alpha + \theta_3 + 180° - \theta_2 = 180°$

$\therefore \alpha = \theta_2 - \theta_3$ //

Ex) Fig 7.13.



phase criterion

$$\theta_2 - (\theta_1 + 90° + \theta_3) = 180°$$

$$\alpha - \theta_1 = 270° \text{ or } -90°$$

$$\therefore \underset{\sim\sim\sim}{\theta_1 = \alpha + 90°}$$ //

$\theta_2 - \theta_3 = \alpha$

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<step 11>. Determine the root locations that satisfy the phase criterion

$$\angle P(s) = 180^\circ$$

<step 12>. Determine the parameter K at a specific root using the mag. criterion.

$$|KP(s)| = 1.$$

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Ex 7.4) 4-th order system

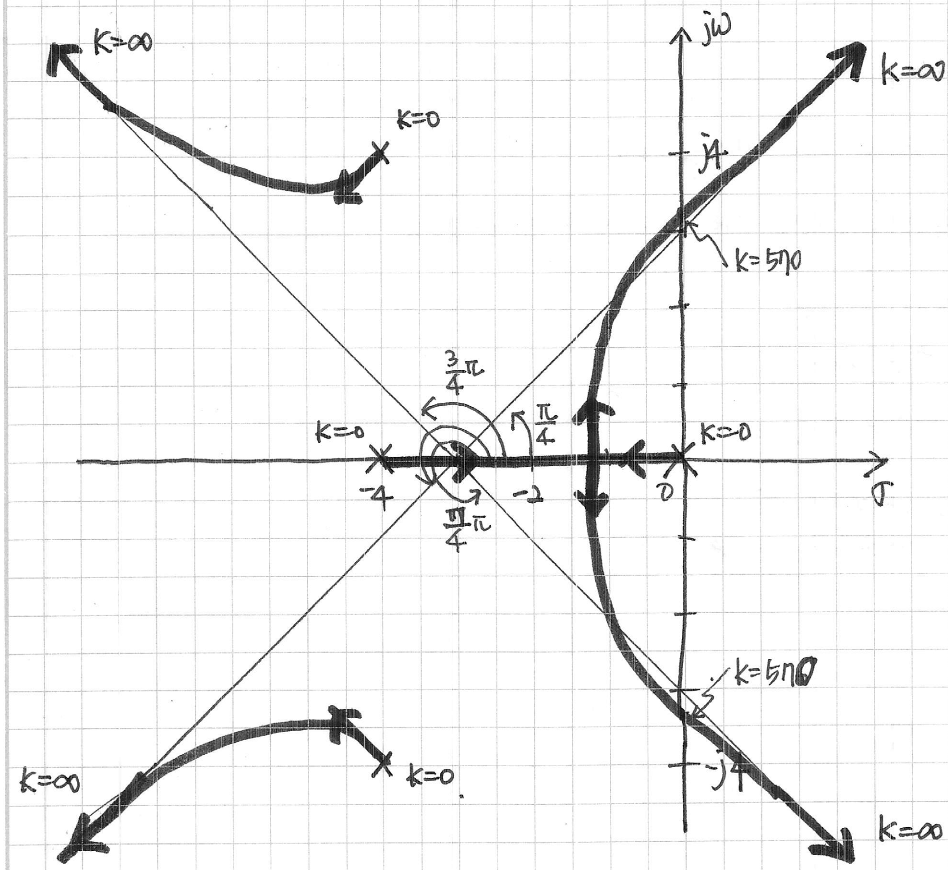
(1). $g(s) = 1 + \dfrac{K}{s^4 + 12s^3 + 64s^2 + 128s} = 0$

교재 (32 → 128 : 수정)

(2) $1 + \dfrac{K}{s(s+4)(s+4+j4)(s+4-j4)} = 0$, $1 + K\,P(s) = 0$.

(3) poles and zeros

(4) ~~Root~~ locus on the real axis



5. # of seperate root loci = max(n_p, n_z) = 4.

6. symmetrical w.r.t real axis

7. asymptotes

$$\sigma_A = \frac{-4 + (-4+j4) + (-4-j4)}{4 - 0} = -3.$$

$$\phi_A = \frac{(2q+1)\pi}{4}, \quad q = 0, 1, 2, 3$$

$$= \frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$$

8. Intersection with imag. axis

$$s^4 + 12s^3 + 64s^2 + 128s + K = 0$$

s^4	1	64	K
s^3	12	128	
s^2	$b_1 = 53.33$	K (if $k=570$)	
s^1	$\boxed{c_1 = 0 \quad 0}$	all zeros.	
s^0	K		

$$b_1 = \frac{12 \times 64 - 128}{12} = \frac{768 - 128}{12} = 53.33$$

$$c_1 = \frac{53.33 \times 128 - 12k}{53.33} .$$

$\therefore$ $53.33 \times 128 > 12k$

$k < 570$

$\therefore$ $0 < K < 570.$

$\therefore$ The locus crosses the imag. axis at $K = 570$.

if $K = 570$, $c_1 = 0$. (any one row are all zeros.)

$\therefore$ Auxiliary eq.

$$53.33\, s^2 + \overset{K}{\overset{\parallel}{570}} = 0$$

$$53.33(s^2 + 10.6) = 0$$

$$\therefore s_{1,2} = \pm j\,3.25$$

* Auxiliary eq.

$$\cancel{53.33\, s^2 + \overset{570}{\overset{\parallel}{K}} = 0}$$

$$\cancel{53.33(s^2 + 10.6) = 0}$$

$$\cancel{\therefore s = \pm j\,3.25}$$

9. breakaway pt. $(-4 \sim 0)$

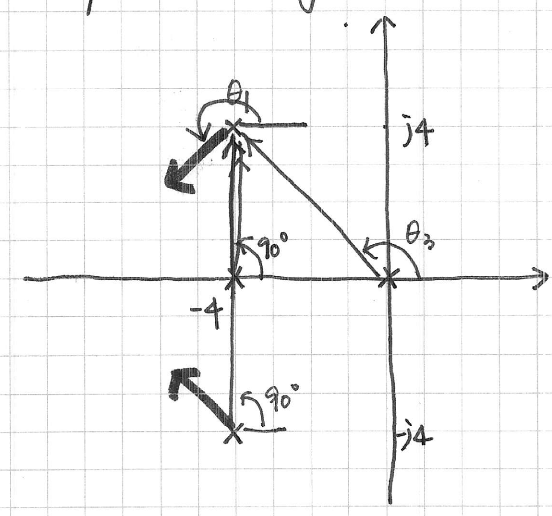
$$K = p(s) = -s(s+4)(s^2+8s+32)$$

Table 1.3

$p(s)$	0	51	68.5	80	85	75	0
s	-4	-3	-2.5	-2	-1.5	-1	0

approximately $s = -1.5$ //, $K = 1.5 \times 2.5 \times (1.5^2 - 8\times1.5 + 32)$ $\doteqdot 83.44$

10. departure angle



$\angle P(s) = 180^\circ$

$\theta_1 + 90^\circ + \theta_3 + 90^\circ = 180^\circ$

$\therefore \theta_1 = -\theta_3 = -135^\circ$ or 225°

11. root loci that satisfy the phase criterion. $\angle p(s) = 180^\circ$

$\angle s_1 + \angle s_1+4 + \angle s_1+4+j4 + \angle s_1+4-j4 = 180^\circ$

12. K at $s = s_i$. $|Kp(s)| = 1$.

(※) 8. K=? at which the locus crosses the imag. axis. (if does so).

$s^4 + 12s^3 + 64s^2 + 128s + K = 0$.

s^4	1	64	K
s^3	12	128	
s^2	$b_1 = 53.33$	K	
s^1	c_1	0	
s^0	K		

$b_1 = \frac{12\times64 - 128}{12} = \frac{768-128}{12} = 53.33$

$c_1 = \frac{53.33\times128 - 12K}{53.33}$

$\therefore$ K > 0 and $53.33\times128 > 12K$

$\therefore$ K ✗ 570. //

$\boxed{0 < K < 570}$

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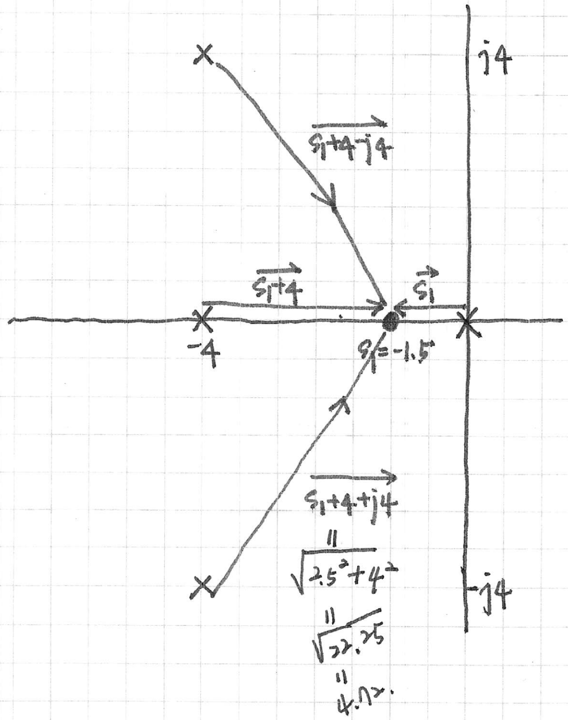
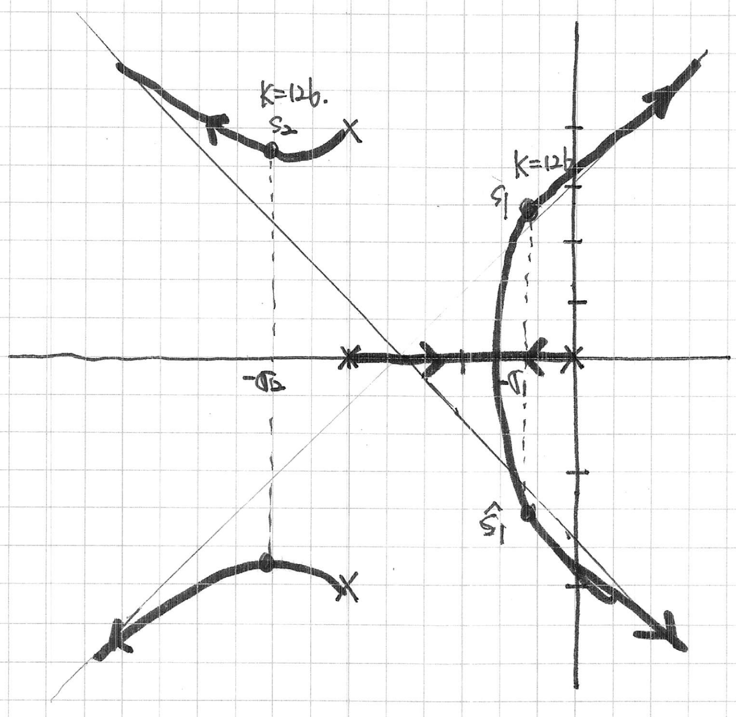
Ex). $s = -1.5$: breakaway point

$$K = |s_1| \cdot |s_1+4| \cdot |s_1+4+j4| \cdot |s_1+4-j4|$$

$$= 1.5 \times 2.5 \times 4.72 \times 4.72 = \underline{83.54}$$



* The complex conjugate roots near the origin are labeled the "dominant roots" of the system because they represent or dominate the transient response.



At k=126.

$$s_{1,2,3,4} = \underbrace{-\sigma_1 \pm j\omega_1}_{\text{dominant pole}},\ -\sigma_2 \pm j\omega_2$$

char. eq. $(s+\sigma_1+j\omega_1)(s+\sigma_1-j\omega_1)(s+\sigma_2+j\omega_2)(s+\sigma_2-j\omega_2)=0.$

~~impulse resp.~~

TF. $$\frac{\bigcirc}{s+\sigma_1+j\omega_1}+\frac{\bigcirc}{s+\sigma_1-j\omega_1}+\frac{\bigcirc}{s+\sigma_2+j\omega_2}+\frac{\bigcirc}{s+\sigma_2-j\omega_2}$$

$$A_1 e^{-\sigma_1 t}\cos(\omega_1 t+\theta_1) \qquad A_2 e^{-\sigma_2 t}\cos(\omega_2 t+\theta_2).$$

dominant.
(과도응답 특성 결정).

빨리 사라짐.

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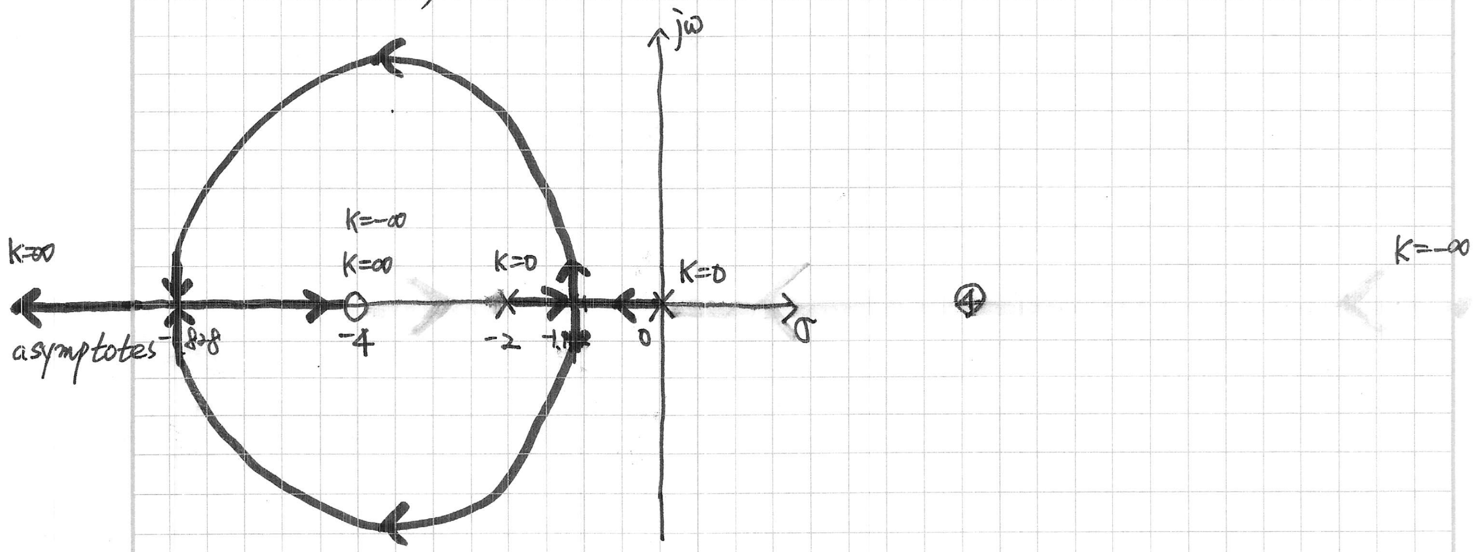
Ex) $g(s) = s(s+2) + K(s+4) = 0$, Find root locus = ?

Sol)

① $1 + \frac{K(s+4)}{s(s+2)} = 0$



② asymptotes

$$\sigma_A = \frac{-2-(-4)}{2-1} = 2$$

$$\phi_A = \frac{(2q+1)\pi}{2-1}, \quad q=0. \qquad \phi_A = 0° \text{ for complementary loci.}$$

$$= \pi,$$

③ breakaway pt : $-2 \sim 0$, $-\infty \sim -4$.

$$K = \frac{-s(s+2)}{s+4} \triangleq A(s)$$

analytically,

$$\frac{dA(s)}{ds} = \frac{-2(s+1)(s+4) + s(s+2)\cdot 1\cdot}{(s+4)^2} = 0$$

$\therefore s^2 + 8s + 8 = 0$

$$s_{1,2} = -4 \pm 2\sqrt{2}$$

$$= -6.828, \ -1.172$$

graphically,

s	-2	-1.5	-1	-0.5	0
A(s)	0	0.3	$\frac{1}{3}$	$\frac{3}{14}$	0

(max)

s	-6	-6.5	-7	-7.5	-8
A(s)	12		11.67		12

(min)

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8. Intersection ~~of~~ with imag. axis.

$$s^2 + (k+2)s + 4k = 0$$

s^2	1	$4k$
s^1	$k+2$	
s^0	$4k$	

$\therefore\ k+2>0 \rightarrow k>-2$

$\quad\ 4k>0 \rightarrow k>0$

$\therefore\ \underline{k>0}$ //

Course ______ Sheet ______ of ______

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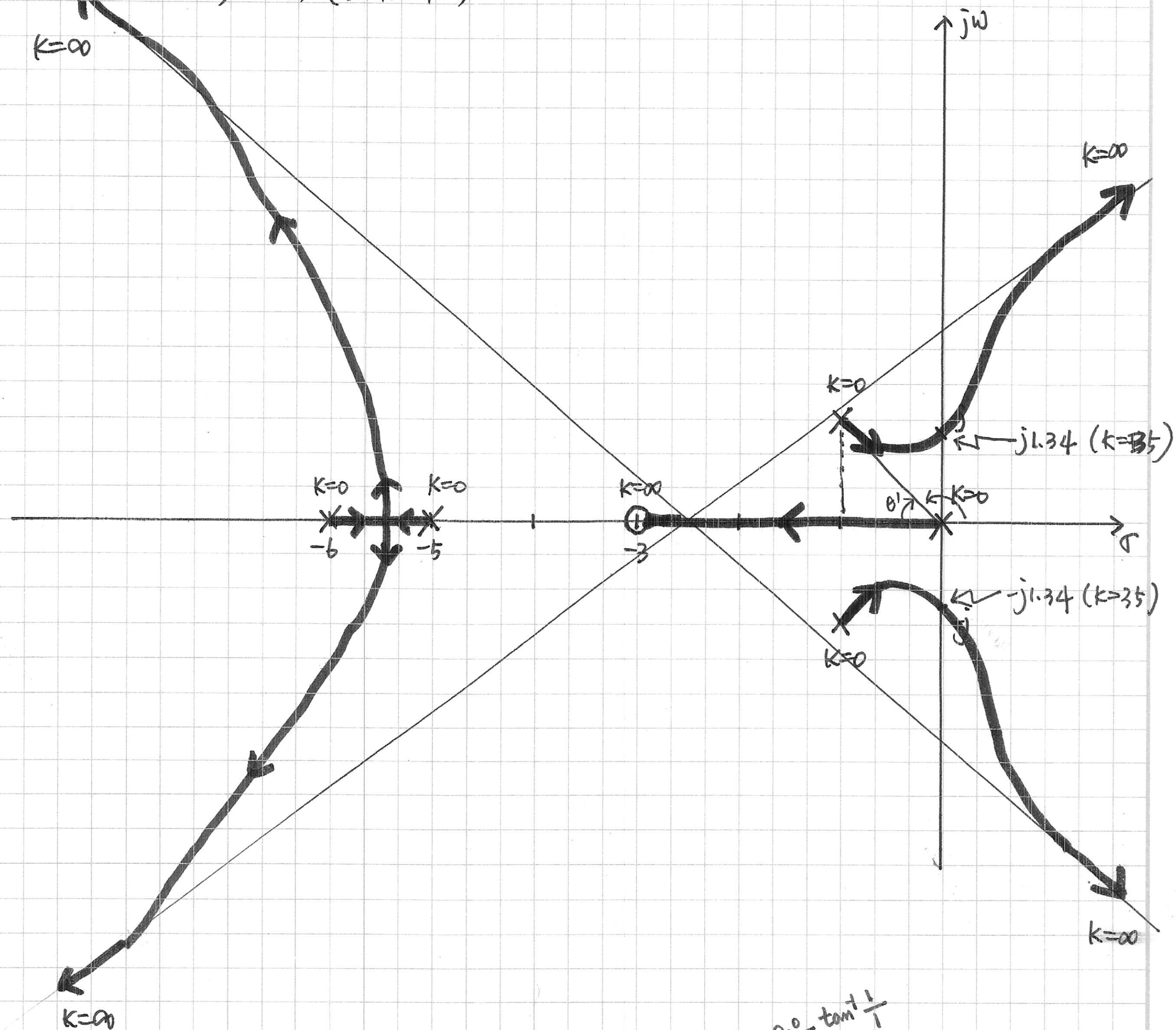
Ex) $s(s+5)(s+6)(s^2+2s+2)+K(s+3)=0$

sol) $1+\dfrac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}=0$



* asymptotes

$$\sigma_A=\frac{-13-(-3)}{5-1}=-2.5$$

$$\phi_A=\frac{(2g+1)\pi}{5-1},\quad g=0,1,2,3$$

$$=\frac{\pi}{4},\ \frac{3}{4}\pi,\ \frac{5}{4}\pi,\ \frac{7}{4}\pi$$

* departure angle. θ

: for infinitesimal pt s_1.

$$\underbrace{\angle s_1+3}_{\substack{\tan^{-1}\frac{1}{2}\\ =26.6^\circ}}-\Big(\underbrace{\angle s_1}_{\substack{180^\circ-\tan^{-1}\frac{1}{1}\\ =135^\circ}}+\underbrace{\angle s_1+1+j}_{=90^\circ}+\theta+\underbrace{\angle s_1+5}_{\substack{\tan^{-1}\frac{1}{4}\\ =14^\circ}}+\underbrace{\angle s_1+6}_{\substack{\tan^{-1}\frac{1}{5}\\ =11.4^\circ}}\Big)=180^\circ$$

$\angle s_1+3=\angle -1+j+3=\angle 2+j\approx 26.6^\circ$

$\angle s_1+5=\angle 4+j\approx 14^\circ$

$\angle s_1+6=\angle 5+j\approx 11.4^\circ$

$\therefore\ \theta=-43.8^\circ$

* Breakaway pt.

* intersection ~~of~~ with the imag. axis.

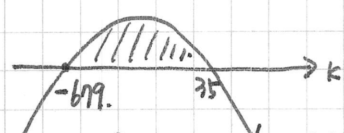
$$s^5 + 13s^4 + 54s^3 + 82s^2 + (60+K)s + 3K = 0$$

s^5	1	54	$60+K$
s^4	13	82	$3K$
s^3	47.7	$60+0.769K$	
s^2	$65.6-0.212K$	$3K$	
s^1	$\dfrac{3940-105K-0.163K^2}{65.6-0.212K}$	0	
s^0	$3K$		

$65.6 - 0.212K > 0 \rightarrow K < 309$

$3940 - 105K - 0.163K^2 > 0 \rightarrow -679 < K < 35$

$\rightarrow K > 0$



$\therefore \; \underline{\underline{0 < K < 35}}$

auxiliary eq.

$$(65.6 - 0.212K)s^2 + 3K = 0 \leftarrow K = 35$$

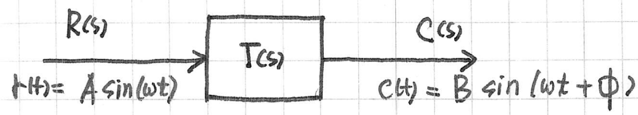
$$58.2s^2 + 105 = 0$$

$$\therefore \; \underline{\underline{s_{1,2} = \pm j1.34}}$$

*. $s = -1$, $K = ?$

$$|KP(s)| = 1 \quad \therefore \; K = \frac{|s|\cdot|s+5|\cdot|s+6|\cdot|s+1+j|\cdot|s+1-j|}{|s+3|} = \frac{1\cdot 4\cdot 5\cdot 1\cdot 1}{2} = 10$$

Chap. 8. Freq. Response. Methods.



The Freq. resp. of a system is defined as the steady-state resp. of the system to sinusoidal input signal as w varies.

Ex) $T(s) = \dfrac{m(s)}{\prod_{i=1}^{n}(s+p_i)}$, $R(s) = \dfrac{Aw}{s^2+w^2}$

$$C(s) = T(s)\cdot R(s) = \frac{k_1}{s+p_1} + \cdots + \frac{k_n}{s+p_n} + \frac{\alpha s+\beta}{s^2+w^2}$$

if the system is stable (At steady state)

$$\lim_{t\to\infty} \mathcal{L}^{-1}\left\{\frac{k_i}{s+p_i}\right\} = \lim_{t\to\infty} k_i e^{-p_i t} = 0 \, , \quad \sim \text{natural resp.} \to 0. \ (\text{transient} \ " \ ")$$

In the steady-state ($t\to\infty$)

$$C_{ss}(s) = \frac{\alpha s+\beta}{s^2+w^2} = \underbrace{\frac{\alpha s+\beta}{Aw}}_{T_{ss}(s).} \cdot \underbrace{\frac{Aw}{s^2+w^2}}_{R(s)}$$

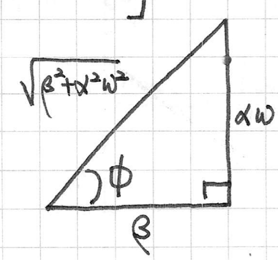
$$T_{ss}(jw) = \frac{jw\alpha+\beta}{Aw} = \underbrace{\frac{\sqrt{\beta^2+w^2\alpha^2}}{Aw}}_{=\,|T_{ss}(jw)|} \underbrace{\angle \tan^{-1}\frac{\alpha w}{\beta}}_{=\,\angle T_{ss}(jw)} = |T_{ss}(jw)|\cdot\angle T_{ss}(jw)$$

$$\therefore\ c(t) = \mathcal{L}^{-1}\left\{\frac{\alpha s+\beta}{s^2+w^2}\right\} = \alpha\cos(wt) + \frac{\beta}{w}\sin(wt)$$

$$= \frac{\sqrt{\beta^2+\alpha^2w^2}}{w}\left[\overbrace{\frac{\beta}{\sqrt{\beta^2+\alpha^2w^2}}}^{\cos(\phi)}\sin(wt) + \overbrace{\frac{\alpha w}{\sqrt{\beta^2+\alpha^2w^2}}}^{\sin\phi}\cos(wt)\right]$$

$$= \frac{\sqrt{\beta^2+\alpha^2w^2}}{w}\sin(wt+\phi)$$

$$= \boxed{A\cdot|T(jw)|\cdot\sin(wt+\phi)} \to f(|T(jw)|, \angle T(jw))$$



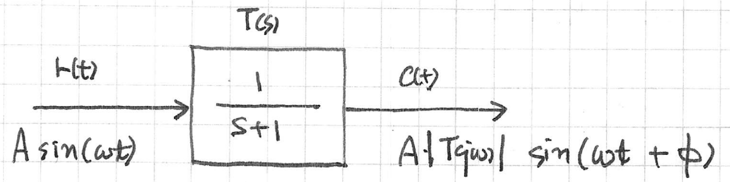
where $|T(jw)| \triangleq \dfrac{\sqrt{\beta^2+\alpha^2w^2}}{Aw}$

$\left(\ \phi = \tan^{-1}\dfrac{\alpha w}{\beta} \triangleq \angle T(jw)\right.$

$\Rightarrow$ We need some graphical plot for $|T(jw)|$ and $\angle T(jw)$.

$s = \sigma + j\omega.$ $\qquad$ $C(s) = T(s) \cdot R(s).$

Ex)



sol)

$$T(j\omega) = \frac{1}{1+j\omega} \Rightarrow |T(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle T(j\omega) = -\tan^{-1}(\omega)$$

if $r(t) = 10\sin(t)$, then $c(t) = \frac{10}{\sqrt{2}}\sin(t - 45^\circ)$

$$\begin{cases} T(j\omega) = 1/\sqrt{2} \\ \angle T(j\omega) = -45^\circ \end{cases}$$

if $r(t) = 10\sin(2t)$, then $c(t) = \frac{10}{\sqrt{5}}\sin(t - 63.43^\circ)$

$$\begin{cases} |T(j\omega)| = 1/\sqrt{5} \\ \angle T(j\omega) = -\tan^{-1}(2) = -63.43^\circ \end{cases}$$

$\vdots$

if $r(t) = 10$ (dc)., then $c(t) = 10.$

$$\begin{cases} |T(0)| = 1 \\ \angle T(j0) = 0^\circ \end{cases}$$

※ $C(j\omega) = T(j\omega) \cdot R(j\omega)$

$$|C(j\omega)| \angle C(j\omega) = |T(j\omega)| \angle T(j\omega) \times |R(j\omega)| \cdot \angle R(j\omega)$$

$$= |T(j\omega)| \cdot |R(j\omega)| \angle T(j\omega) + R(j\omega)$$

~~⇒ We need some graphical plots for the $|T(j\omega)|$ and $\angle T(j\omega)$~~.

8.2 Freq. response plots.

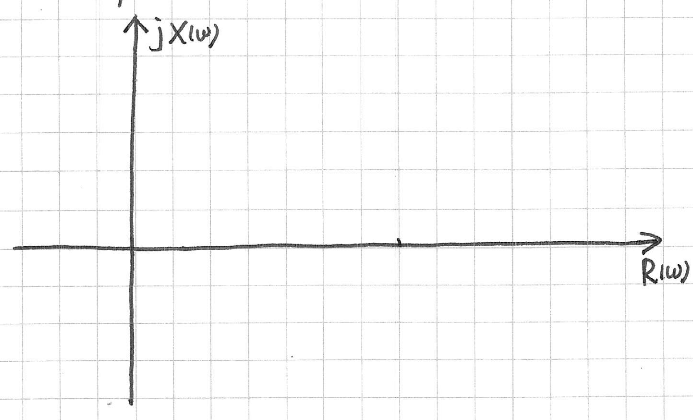
$$G(j\omega) = G(s)\Big|_{s=j\omega} = R(\omega) + jX(\omega)$$

$$= |G(j\omega)| \angle G(j\omega) \overset{\text{or}}{=} |G(j\omega)| e^{j\phi(\omega)}$$

where $|G(j\omega)| = \sqrt{R^2(\omega) + X^2(\omega)}$

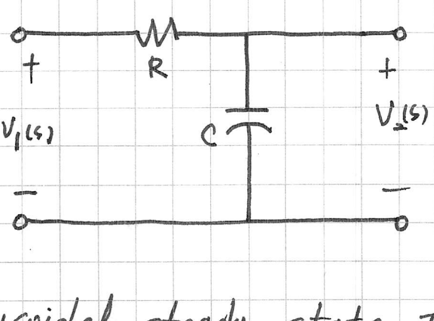
$$\angle G(j\omega) = \tan^{-1} \frac{X(\omega)}{R(\omega)}$$

<polar plot>



: polar plane
$G(j\omega)$ plot as ω varies.

Ex 8.1) freq. response of a RC filter.



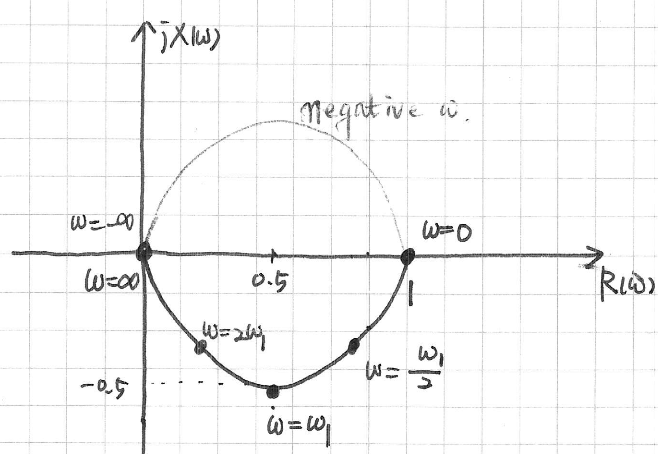
$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{RCs+1}$$

sinusoidal steady-state TF.

$$G(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{1 + j\frac{\omega}{\omega_1}}, \quad \omega_1 = \frac{1}{RC} > 0$$

$$= \frac{1 - j\frac{\omega}{\omega_1}}{1 + \left(\frac{\omega}{\omega_1}\right)^2} = \underbrace{\frac{1}{1 + \left(\frac{\omega}{\omega_1}\right)^2}}_{R(\omega)} - j\underbrace{\left(\frac{\frac{\omega}{\omega_1}}{1 + \left(\frac{\omega}{\omega_1}\right)^2}\right)}_{X(\omega)}$$

ω	0	$0.5\omega_1$	ω_1	$2\omega_1$	∞
$R(\omega)$	1	0.8	0.5	0.2	0
$X(\omega)$	0	-0.4	-0.5	-0.4	0



(Another method)

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

① $$|G(j\omega)| = \frac{1}{|1+j\omega/\omega_1|} = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}}$$

② $$\angle G(j\omega) = -\angle 1+j\frac{\omega}{\omega_1} = -\tan^{-1}\frac{\omega}{\omega_1}$$

ω	0	$\frac{\omega_1}{2}$	ω_1	$2\omega_1$	∞
$\lvert G(j\omega)\rvert$	1	$2/\sqrt{5}$	$1/\sqrt{2}$	$1/\sqrt{5}$	0
$\angle\phi(\omega)$	$0°$	$-26.6°$	$-45°$	$-63°$	$-90°$

* ~~dc input ($\omega=0$), $V_2(s) = A\sin\omega t$.~~

~~$\omega=0$ (dc input) $V_1(s)=A \rightarrow V_2(s) =$~~

$\omega=\omega_1$. ($V_1(t) = A\sin\omega_1 t \rightarrow V_2(t) = \frac{A}{\sqrt{2}}\sin(\omega_1 t - 45°)$

Ex 8.2) polar plot of a T.F.

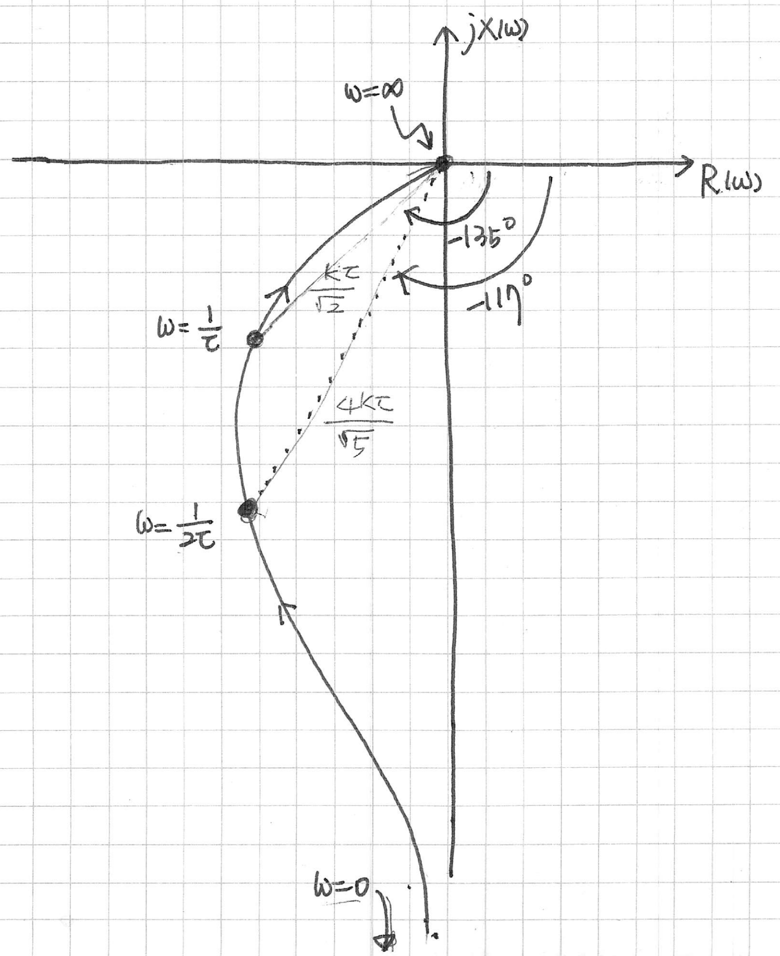
$$\cancel{G(j\omega)} \; G(s) = \frac{K}{s(s\tau+1)}$$

sol) $G(j\omega) = \dfrac{K}{j\omega(j\omega\tau+1)} = \dfrac{K}{j\omega - \omega^2\tau}$

$$|G(j\omega)| = \frac{K}{|j\omega - \omega^2\tau|} = \frac{K}{\sqrt{\omega^2+\omega^4\tau^2}} = \frac{K}{\omega\sqrt{1+\omega^2\tau^2}}$$

$$\phi(\omega) = -\tan^{-1}\left(\frac{\omega}{-\omega^2\tau}\right) = -\cancel{\tan^{-1}\frac{1}{-\omega\tau}} - \tan^{-1}\frac{1}{-\omega\tau} \;\; "$$

ω	0	$\frac{1}{2\tau}$	$\frac{1}{\tau}$	$\frac{2}{\tau}$	∞
$\lvert G(j\omega)\rvert$	∞	$\frac{4k\tau}{\sqrt{5}}$	$\frac{k\tau}{\sqrt{2}}$	$\longrightarrow$	0
$\phi(\omega)$	-90°	-117°	-135°	$\longrightarrow$	-180°



<Bode plots> ~ logarithmic plots.

(magnitude plot : $20\log_{10}|G(j\omega)|$ versus ω. (dB)
(phase plot : $\phi(\omega)$ versus ω.

Ex 8.3) Bode diagram of a RC filter. ~ Ex 8.1)

$$G(j\omega) = \frac{1}{j\omega RC+1} = \frac{1}{j\omega\tau+1}, \quad \tau \triangleq RC : \text{time constant.}$$

*. Magnitude plot.

$$20\log|G| = 20\log\frac{1}{|j\omega\tau+1|} = 20\log(1) - 20\log|j\omega\tau+1|$$

$$= -20\log(1+\omega^2\tau^2)^{1/2}$$

$$= -10\log(1+\omega^2\tau^2)$$ //

ω	0	$\frac{1}{2\tau}$	$\frac{1}{\tau}$	$\frac{3}{2\tau}$	$\frac{2}{\tau}$	$\frac{5}{2\tau}$
$20\log\lvert G\rvert$	0	$-10\log(\frac{5}{4})$	$-3.01 = -10\log 2$.	$-10\log(\frac{9}{4})$	$-7 \simeq -10\log 5$	

~ Fig 8.6 (a).

* phase plot.

$$\phi(\omega) = \angle G(j\omega) = -\angle j\omega\tau+1 = -\tan^{-1}\frac{\omega\tau}{1}$$ //

ω	0	$\frac{1}{2\tau}$	$\frac{1}{\tau}$	$\frac{3}{2\tau}$	$\frac{2}{\tau}$	$\frac{5}{2\tau}$	∞
$\phi(\omega)$	0°	-27°	-45°		-63°		-90°.

~ Fig 8.6 (b).

⇒ A linear scale of freq. is not judicious choice.

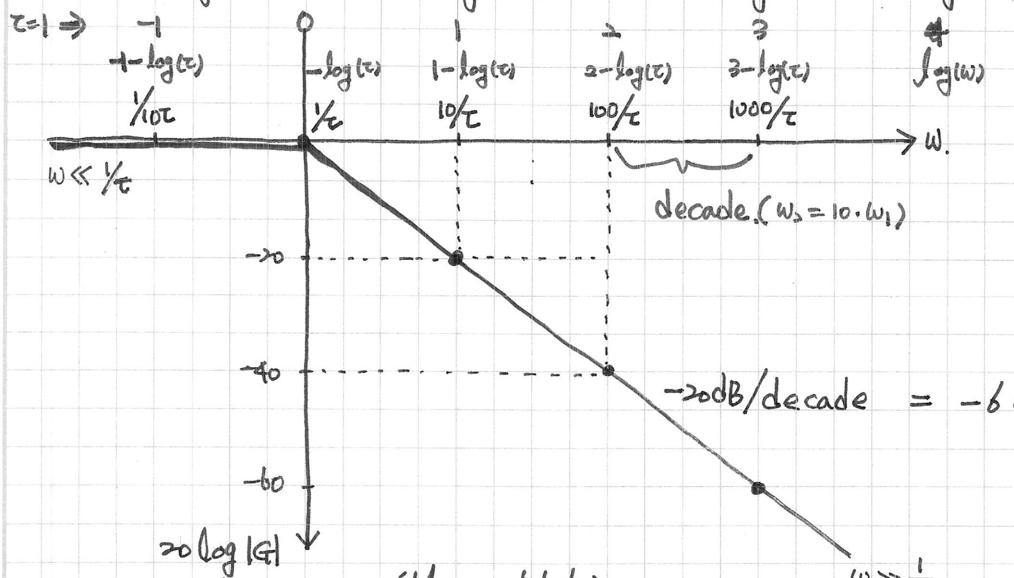
⇒ logarithmic scale of freq.

For small freq. $\omega \ll \frac{1}{\tau}$,

$$20\log|G| \cong -10\log(1) = 0\,dB \quad \leftarrow -10\log(1+\cancel{\omega^2\tau^2})$$

For large freq. $\omega \gg \frac{1}{\tau}$

$$20\log|G| = -20\log(\omega\tau) = -20\log(\omega) - 20\log(\tau).$$



<Mag. plot>.

*. decade : freq. interval btn ω_1 and $\omega_2 (= 10\cdot\omega_1)$
octave : 〃 〃 $(= 2\omega_1)$

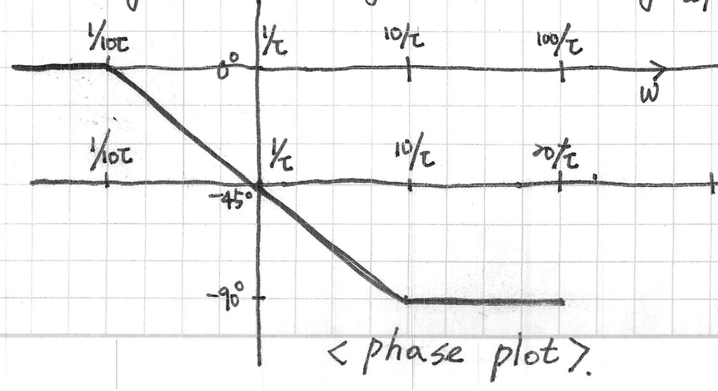
For $\omega \gg \frac{1}{\tau}$, the gain dif. over a decade $(\omega_2 = 10\omega_1)$,

$$20\log|G(\omega_1)| - 20\log|G(\omega_2)| = -20\log(\omega_1\tau) - (-20\log(\omega_2\tau))$$

$$= 20\log\frac{\omega_2\tau}{\omega_1\tau} = 20\log(10) = 20\,dB$$

Also, over an octave $(\omega_2 = 2\cdot\omega_1)$

$$20\log|G(\omega_1)| - 20\log|G(\omega_2)| = 20\log\frac{\omega_2\tau}{\omega_1\tau} = 20\log(2) = 6.02\,dB.$$



$\omega = \frac{1}{\tau}$, $\phi(\omega) = -\tan^{-1}(\omega\tau) = -45°$

$\omega \ll \frac{1}{\tau}$, $\phi(\omega) = 0°$

$\omega \gg \frac{1}{\tau}$, $\phi(\omega) = -90°$

<phase plot>.

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* Generalized TF

$$G(j\omega) = \frac{K_b \cdot \prod_{i=1}^{Q} (1+j\omega\tau_i)}{(j\omega)^N \prod_{m=1}^{M} (1+j\omega\tau_m) \prod_{k=1}^{R} \left[1+\frac{2\xi_k}{\omega_{nk}} j\omega + \left(\frac{j\omega}{\omega_{nk}}\right)^2\right]}$$

$\prod \rightarrow \sum$

: 대수눈금의 장점.

$$\Rightarrow 20\log|G| = 20\log K_b + 20\sum_{i=1}^{Q} \log|1+j\omega\tau_i|$$
$$- 20\log|(j\omega)^N| - 20\sum_{m=1}^{M} \log|1+j\omega\tau_m|$$
$$- 20\sum_{k=1}^{R} \log\left|1+\frac{2\xi_k}{\omega_{nk}} j\omega + \left(\frac{j\omega}{\omega_{nk}}\right)^2\right|$$

$$\phi(\omega) = \sum_{i=1}^{Q} \tan^{-1}(\omega\tau_i) - N(90^\circ) - \sum_{m=1}^{M} \tan^{-1}\omega\tau_m$$
$$- \sum_{k=1}^{R} \tan^{-1}\left(\frac{2\xi_k \omega_{nk}\omega}{\omega_{nk}^2 - \omega^2}\right)$$

① Constant gain K_b

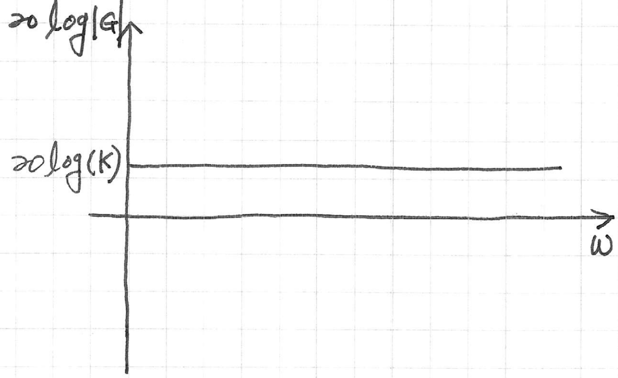
② Poles (or zeros) at the origin $(j\omega)$

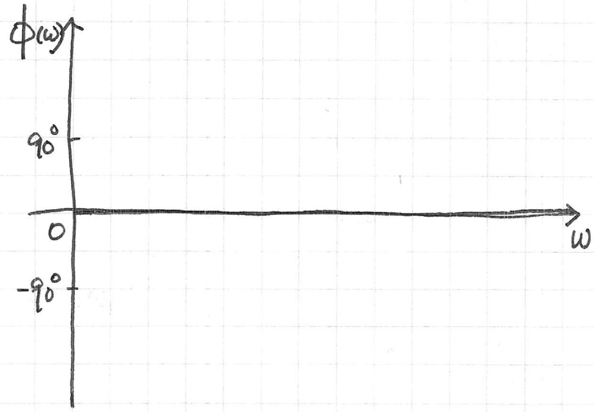
③ 〃 on the real axis $(j\omega\tau+1)$

④ complex conjugate poles (or zeros) $\left[1+\frac{2\xi}{\omega_n} j\omega + \left(\frac{j\omega}{\omega_n}\right)^2\right]$

① Constant gain K_b.

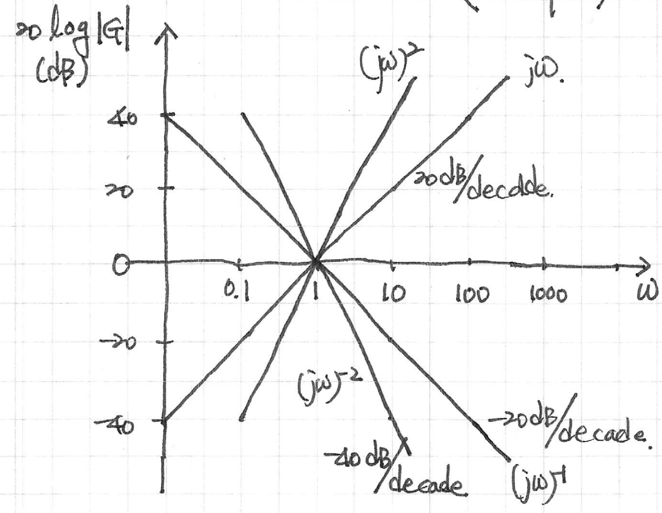
$G(j\omega) = K_b. \Rightarrow \begin{cases} 20\log|G| = \text{costant (dB)} \\ \phi(\omega) = 0^\circ \end{cases}$

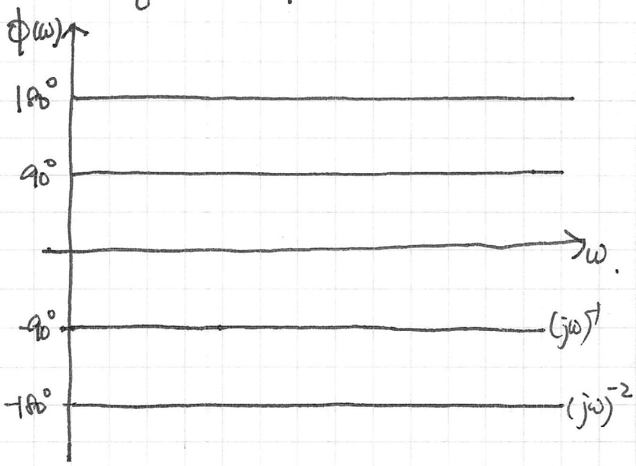




② Poles at the origin

$G(j\omega) = \dfrac{1}{j\omega} \Rightarrow \begin{cases} 20\log|G| = -20\log(\omega) \text{ (dB.)} \\ \phi(\omega) = -\tan^{-1}\dfrac{\omega}{0} = -90^\circ \end{cases}$





$G(j\omega) = \dfrac{1}{(j\omega)^{2}_{N}} \Rightarrow \begin{cases} 20\log|G| = -20\cdot \overset{N}{2}\log(\omega). \\ \phi(\omega) = -\tan^{-1}\dfrac{0}{-\omega^2} = -180^\circ = -90^\circ \times 2_{N}. \end{cases}$

$= \dfrac{1}{-\omega^2}$

*(cf) zeros at the origin.

$G(j\omega) = j\omega \Rightarrow \begin{cases} 20\log|G| = 20\log(\omega). \\ \phi(\omega) = \tan^{-1}\dfrac{\omega}{0} = 90^\circ \end{cases}$

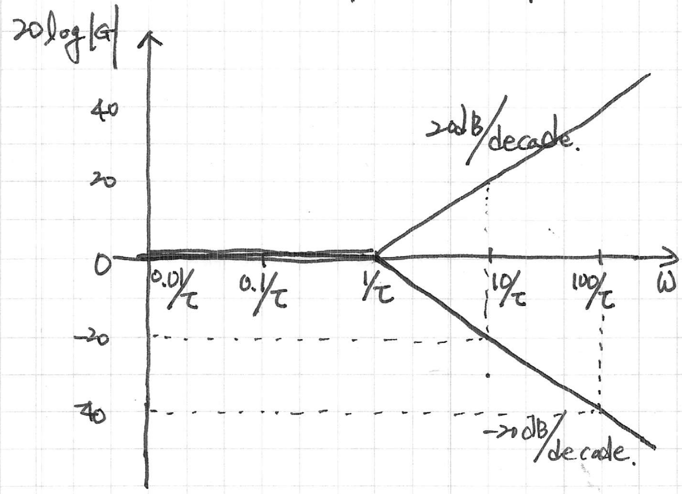
$G(j\omega) = (j\omega)^2 \Rightarrow \begin{cases} 20\log|G| = 40\log(\omega) \\ \phi(\omega) = \tan^{-1}\dfrac{0}{-\omega^2} = 180^\circ \end{cases}$

③ poles on the real axis

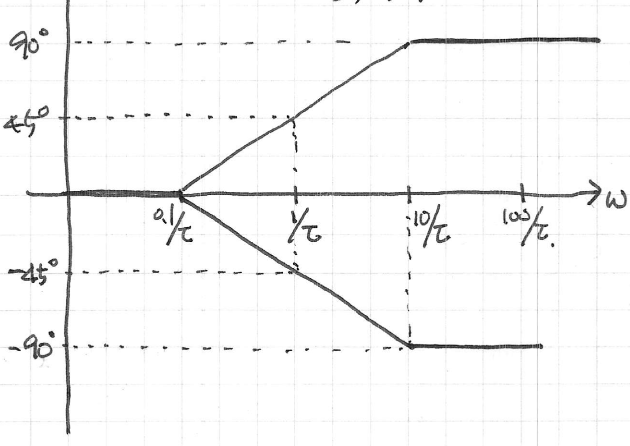
$$G(j\omega) = \frac{1}{1+j\omega\tau} \Rightarrow \begin{cases} 20\log|G| = -10\log(1+\omega^2\tau^2) \\ \phi(\omega) = -\tan^{-1}(\omega\tau) \end{cases}$$

For $\omega \ll \frac{1}{\tau}$, $\begin{cases} 20\log|G| \Rightarrow 0\,dB. \\ \phi(\omega) \Rightarrow 0^\circ \end{cases}$

For $\omega \gg \frac{1}{\tau}$ $\begin{cases} 20\log|G| \Rightarrow -20\log(\omega\tau) = -20\log(\omega) - 20\log(\tau). \\ \phi(\omega) \Rightarrow -90^\circ \end{cases}$



At $\omega = \frac{1}{\tau}$, $\phi(\omega) = -\tan^{-1}(1) = -45^\circ$



$\omega = \frac{1}{\tau}$: break freq. $\Leftarrow 20\log|G| = -10\log(2) = -3.02\,dB$

i.e. lower 3dB freq.

* zeros on the real axis.

$$G(j\omega) = 1+j\omega\tau \Rightarrow \begin{cases} 20\log|G| = +10\log(1+\omega^2\tau^2) \\ \phi(\omega) = \tan^{-1}(\omega\tau). \end{cases}$$

Table 8.3 ~ Summary

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④ complex conjugate poles

$$G(j\omega) = \frac{1}{1 + j\frac{2\xi\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2} \leftarrow \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{1}{1 + j2\xi u - u^2}, \quad u \triangleq \frac{\omega}{\omega_n} \text{ : normalized freq.}$$

$$\Rightarrow 20\log|G| = -20\log|1 + j2\xi u - u^2|$$

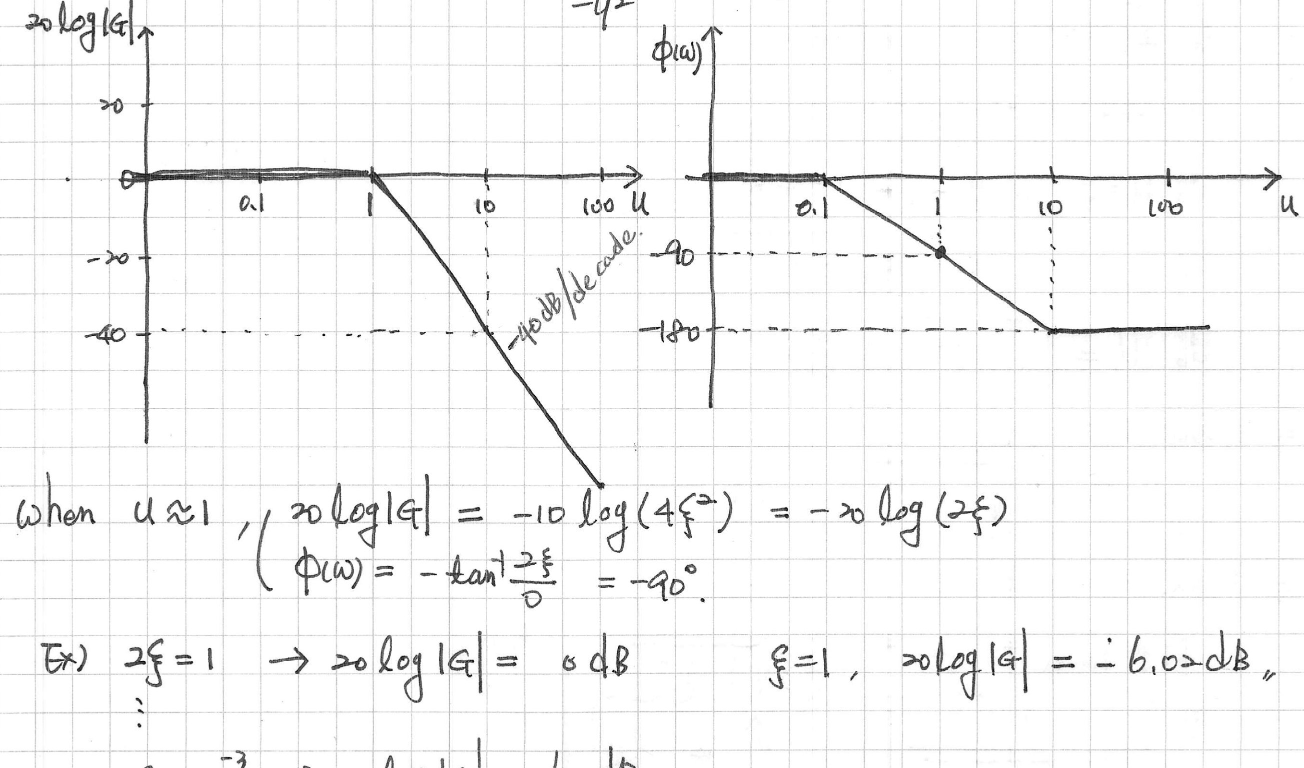
$$= -10\log\left((1-u^2)^2 + 4\xi^2u^2\right)$$

$$\phi(\omega) = -\tan^{-1}\frac{2\xi u}{1-u^2}$$

when $u \ll 1$, $\begin{cases} 20\log|G| \cong -10\log(1) = 0\,dB \\ \phi(\omega) \cong -\tan^{-1}\frac{0}{1} = 0^\circ \end{cases}$

when $u \gg 1$, $20\log|G| \cong -10\log(u^4) = -40\log(u)$

$\phi(\omega) \cong -\tan^{-1}\frac{2\xi u}{-u^2} \approx -180^\circ$



when $u \approx 1$, $\begin{cases} 20\log|G| = -10\log(4\xi^2) = -20\log(2\xi) \\ \phi(\omega) = -\tan^{-1}\frac{2\xi}{0} = -90^\circ. \end{cases}$

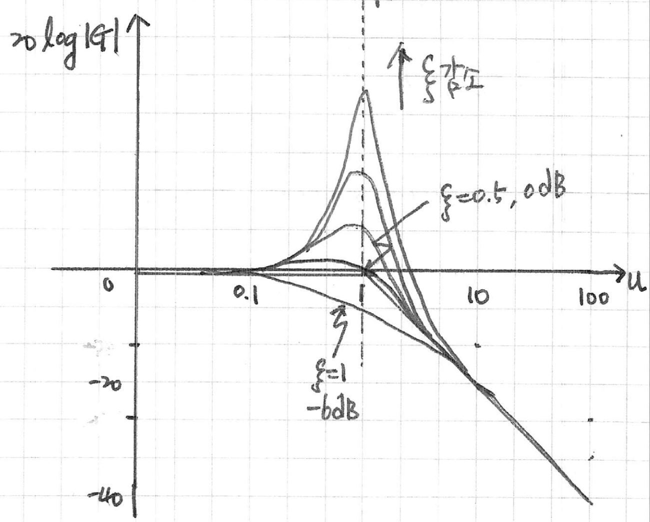
Ex) $2\xi = 1 \rightarrow 20\log|G| = 0\,dB$ $\qquad \xi = 1, \; 20\log|G| = -6.02\,dB$ //

$\vdots$

$2\xi = 10^{-3} \rightarrow 20\log|G| = 60\,dB$

$2\xi = 10^{-5} \rightarrow 20\log|G| = 100\,dB$ //

Actual curve



※ resonant freq. (when $\xi < 0.707$)

$$\frac{d|G(j\omega)|}{du} = 0 \quad , \quad |G(j\omega)| = \frac{1}{|1 + j2\xi u - u^2|} = \frac{1}{\sqrt{(1-u^2)^2 + 4\xi^2u^2}}$$

$$\frac{d|G(j\omega)|}{du} = \frac{-\frac{1}{2}\left[\overset{\neq 0}{(1-u^2)^2 + 4\xi^2u^2}\right]^{-1/2}\left[2(1-u^2)(-2u) + 8\xi^2 u\right]}{(1-u^2)^2 + 4\xi^2u^2} = 0$$

$$\therefore\ 2(1-u^2)(-2u) + 8\xi^2 u = 0.$$

$$4u(u^2 - 1 + 2\xi^2) = 0$$

$$\therefore\ u = 0 \ \text{or} \ u^2 - 1 + 2\xi^2 = 0$$

$$u = \pm\sqrt{1 - 2\xi^2}$$

$$\therefore\ u = \sqrt{1-2\xi^2} \triangleq \frac{\omega_r}{\omega_n}$$

⇒ resonant freq

$$\omega_r \triangleq \omega_n\sqrt{1-2\xi^2} \qquad (8.36)$$

⇒ Max. magnitude.

$$|G(j\omega)|_{max} = M_{p\omega} = \frac{1}{2\xi\sqrt{1-\xi^2}} \qquad (8.37)$$

$$\because)\ \left.\frac{1}{\sqrt{(1-u^2)^2 + 4\xi^2u^2}}\right|_{u=\sqrt{1-2\xi^2}} =$$

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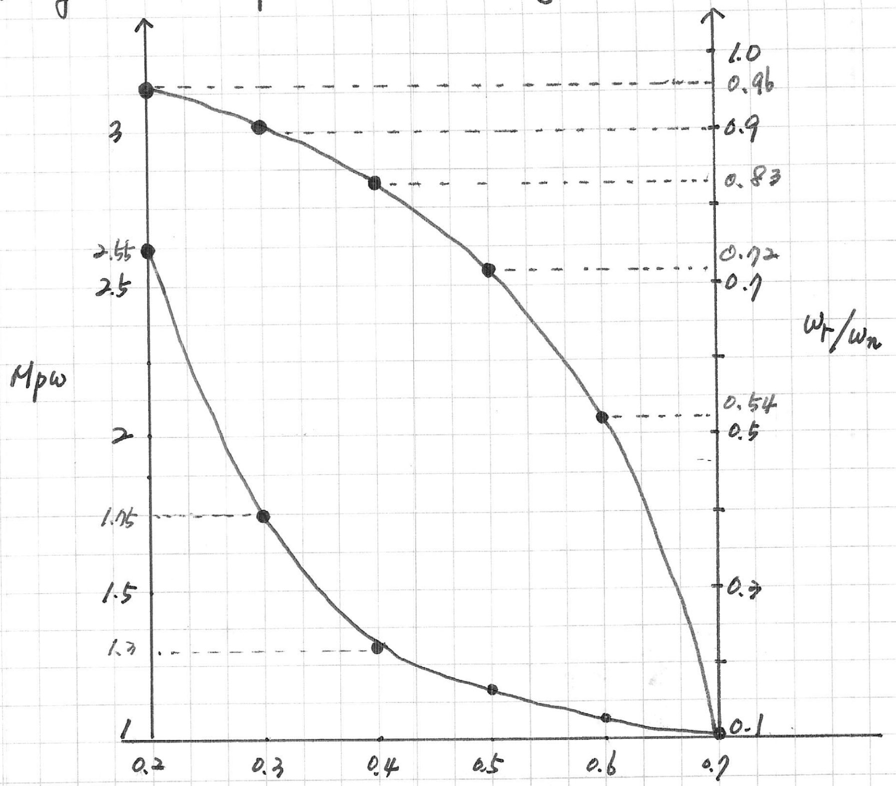
※ Fig 8.11 ~ $M_{p\omega}$ and ω_r v.s. ξ.

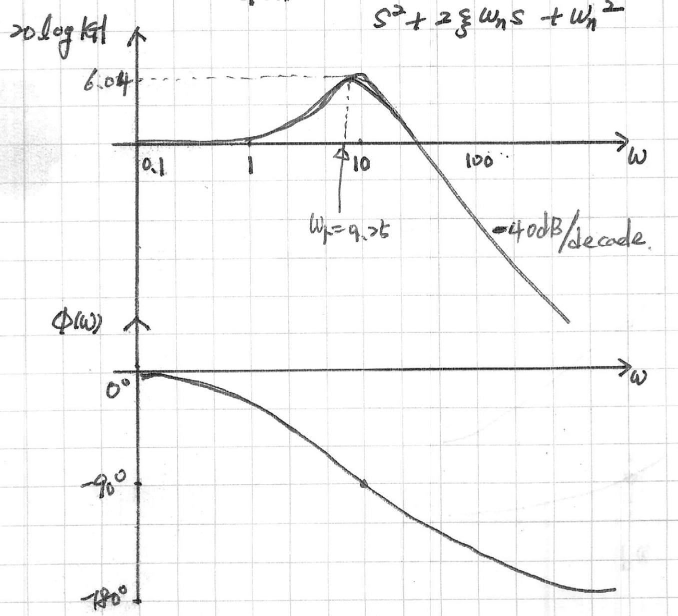


experimentally determined freq. resp ($M_{p\omega}$, ω_r) → ξ estimation. (ω_n 〃)

∴ Transfer fn.

Ex) Given Bode plot of a 2nd order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$



$20\log(M_{p\omega}) = 6.04\,dB$.

$\therefore M_{p\omega} = 2$

⇒ From Fig 8.11..

$\xi = 0.26$.

$\Rightarrow \frac{\omega_r}{\omega_n} = 0.925$

$\therefore \omega_n = \frac{\omega_r}{0.925} \cong 10$

$$\Rightarrow G(s) = \frac{100}{s^2 + 5.2s + 100}$$ 〃

※ Graphical method.

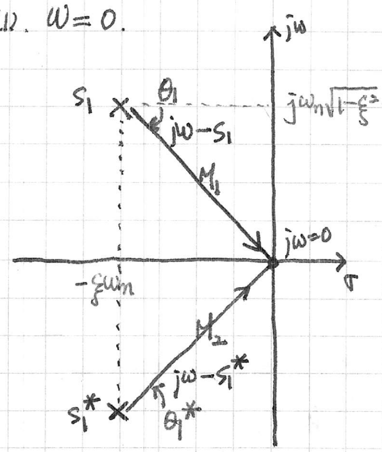
$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \quad \Rightarrow \quad G(j\omega) = \left.\frac{\omega_n^2}{(s-s_1)(s-s_1^*)}\right|_{s=j\omega}$$

$$= \frac{\omega_n^2}{(j\omega - s_1)(j\omega - s_1^*)}$$

$$|G(j\omega)| = \frac{\omega_n^2}{|j\omega - s_1| \cdot |j\omega - s_1^*|}$$

$$\phi(\omega) = -\angle j\omega - s_1 - \angle j\omega - s_1^*$$

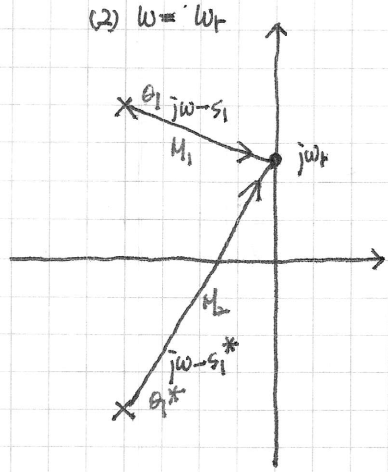
(1). $\omega = 0$.



$$|G| = \frac{\omega_n^2}{\omega_n \cdot \omega_n} = 1.$$

$$\phi = -(+\theta_1) + \theta_1^* = 0°$$

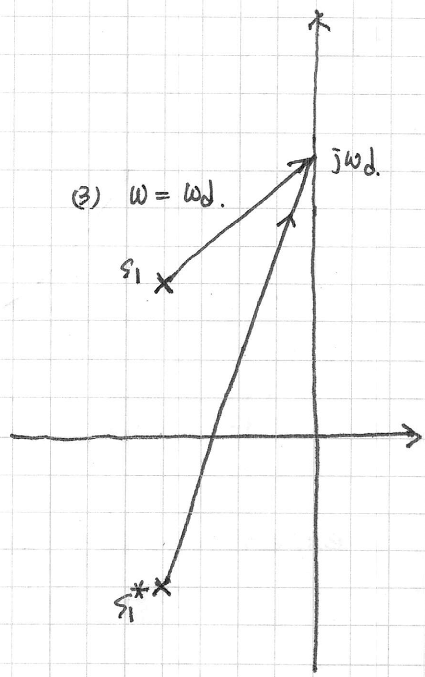
(2) $\omega = \omega_r$



$$|G| = \frac{\omega_n^2}{M_1 \cdot M_2}$$

$$\phi = -\theta_1 - \theta_1^*.$$

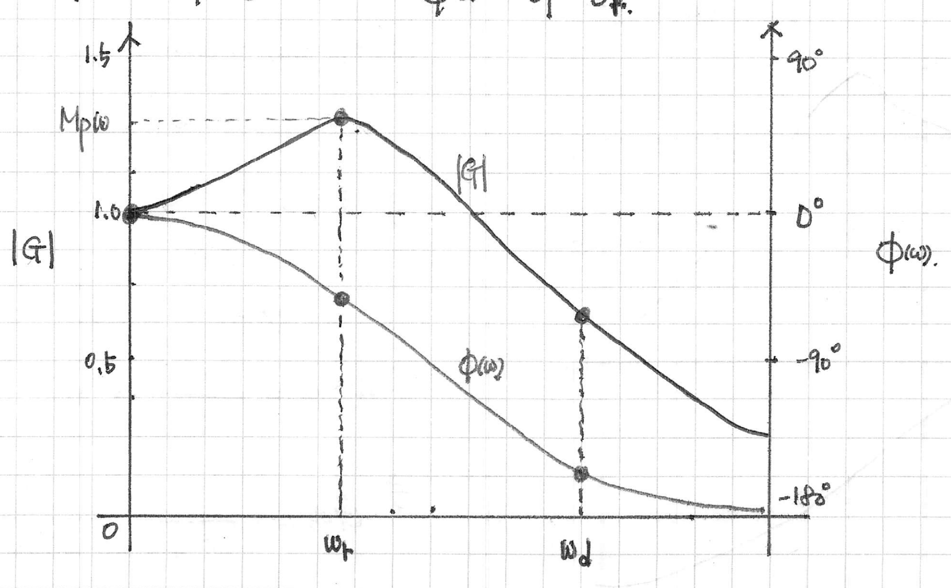
(3) $\omega = \omega_d$.



$*$ $\omega = \infty$.

$$|G| = \frac{\omega_n^2}{\infty} = 0.$$

$$\phi = -90° - 90° = -180°$$



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* "minimum phase" TF if all its zeros lie in the lhp.

"non-minimum phase" TF if it has zeros in the rhp.

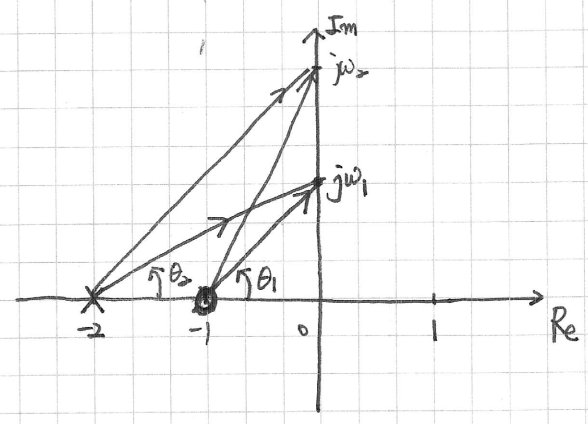
Ex) $G_1(s) = \dfrac{s+1}{s+2}$, $G_2(s) = \dfrac{s-1}{s+2}$

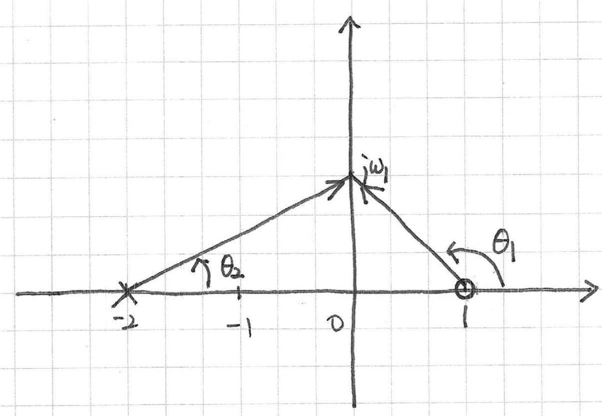
$\Rightarrow G_1(j\omega) = \dfrac{1+j\omega}{2+j\omega}$ $\quad G_2(j\omega) = \dfrac{-1+j\omega}{2+j\omega}$

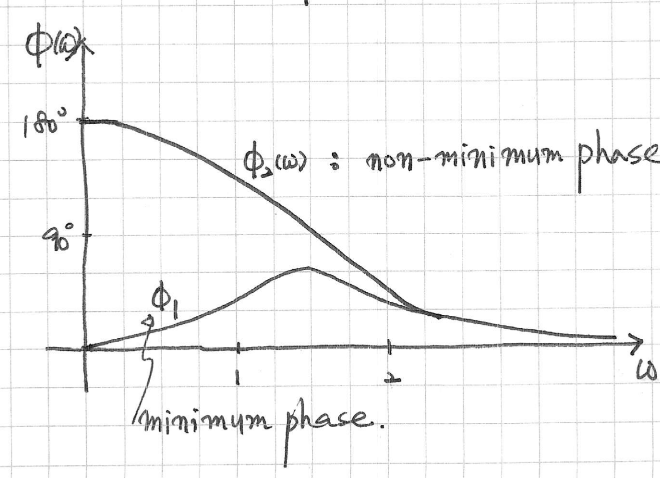
$$\begin{cases} 20\log|G_1| = 10\log\left(\dfrac{1+\omega^2}{4+\omega^2}\right) \\ 20\log|G_2| = 10\log\left(\dfrac{1+\omega^2}{4+\omega^2}\right) \end{cases}$$

$\Rightarrow$ same magnitude characteristics. $|G_1| = |G_2|$

$$\begin{cases} \phi_1(\omega) = \tan^{-1}\left(\dfrac{\omega}{1}\right) - \tan^{-1}\left(\dfrac{\omega}{2}\right) = \theta_1 - \theta_2 \\ \phi_2(\omega) = \tan^{-1}\left(\dfrac{\omega}{-1}\right) - \tan^{-1}\left(\dfrac{\omega}{2}\right) = \theta_1 - \theta_2 \end{cases}$$







net phase shift over all freq. range (0 ~ ∞) : minimum phase TF ≪ non-minimum phase TF

≈ small $\qquad$ ≈ 180°

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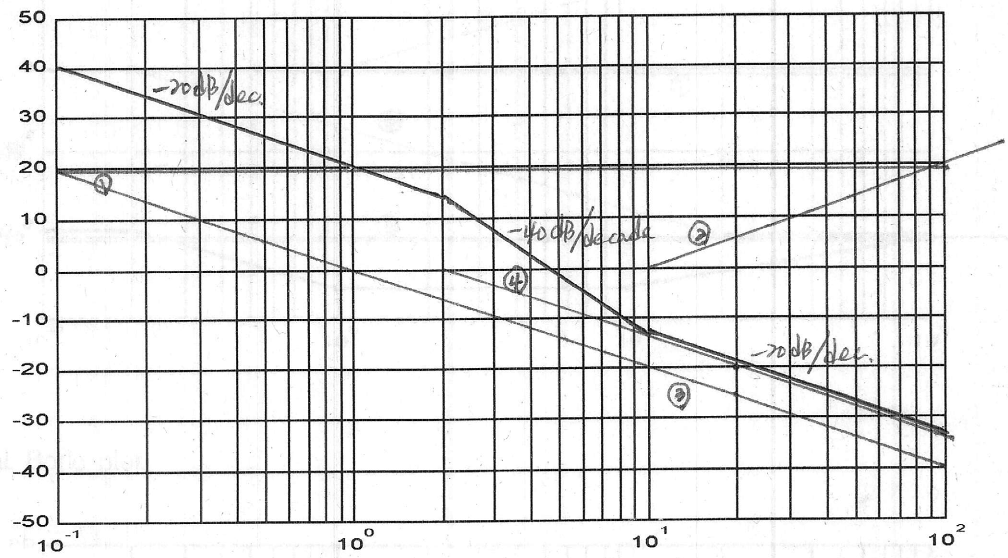
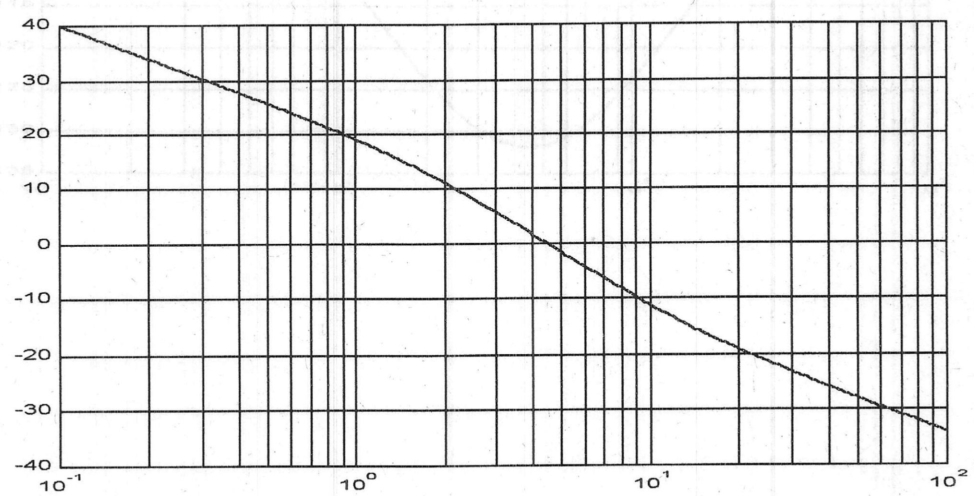
* $G_3(s) = \dfrac{s+2}{s+1}$: phase-lag

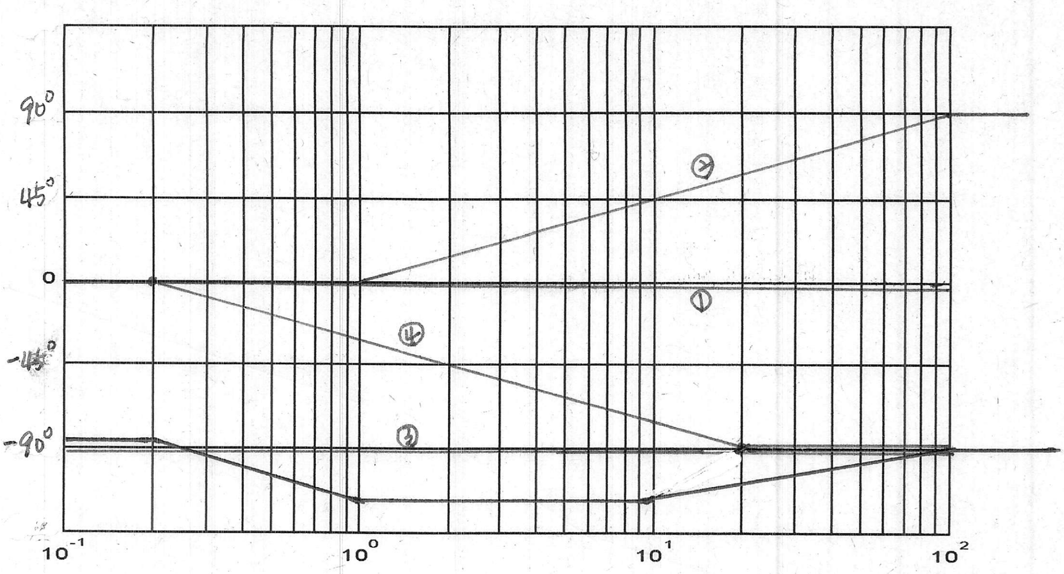
8.3 Example of Drawing the Bode Diagram

Ex1) $G(jw) = \dfrac{10\,(1+j0.1w)}{jw\,(1+j0.5w)} = \underset{①}{10} \times \underset{②}{(1+jw/10)} \times \underset{③}{\dfrac{1}{jw}} \times \underset{④}{\dfrac{1}{1+jw/2}}$

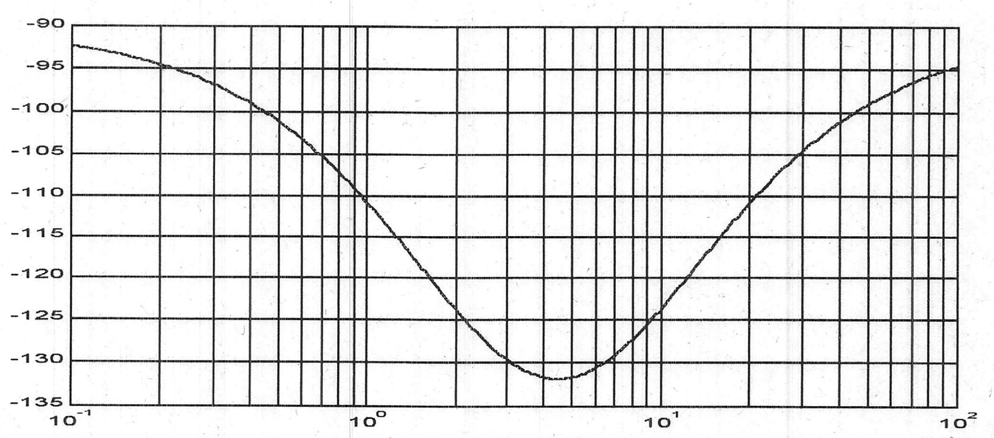


Compare actual Bode plot





Compare actual Bode plot



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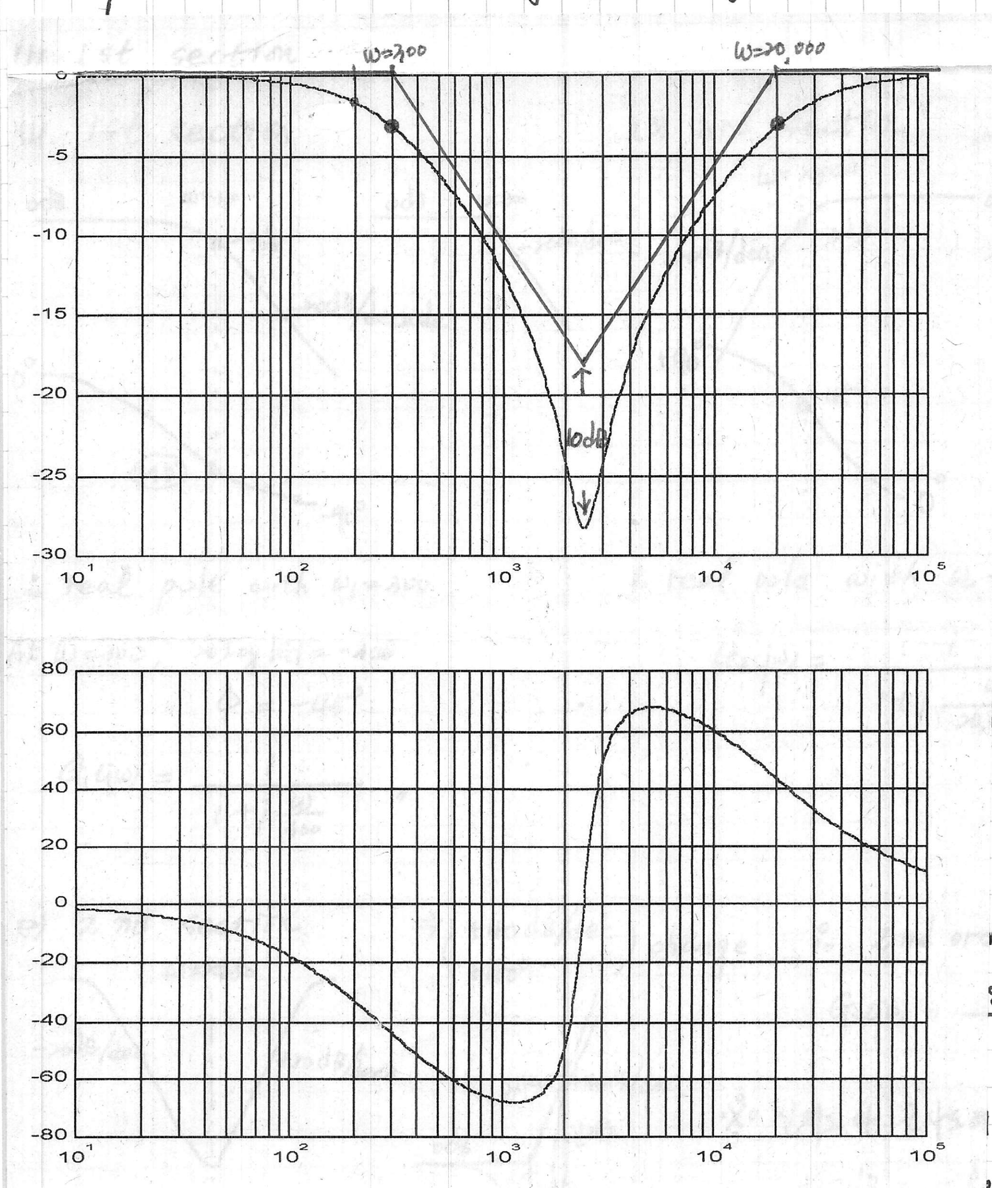
8.4 Frequency Response Measurements.

Signal analyzer

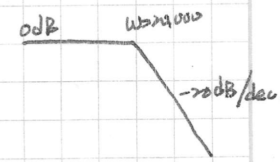
↓

Freq. resp. for unknown system ⟶ Transfer fn.
(Bode diagram)

Example) Given Bode diagram ~ Fig 8.23.



dB



= 20,000

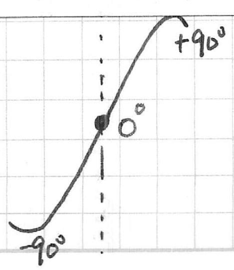
0 / 000 "

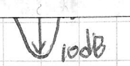
ler zeros. ($\omega_n = 2450$)

$$\frac{s^2 + 2\xi\omega_n s + \omega_n^2}{\omega_n^2}$$

차이.

g($M_{p\omega}$) ∴ $M_{p\omega} = \sqrt{10}$.





$$M_{p\omega} = \sqrt{10} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$ From (eq. 8.37)

$\xi^2 = 0.974$ or 0.0256

∴ $\xi = \cancel{0.988}$ or 0.16 "

$\xi < 0.707$

※ 2차 영점의 실제도와 근사도의 차이

$$+10\,dB = 20\log(M_{p\omega})$$

$$\therefore M_{p\omega} = \sqrt{10} = \frac{1}{2\xi\sqrt{1-\xi^2}} \quad \text{from Eq. (8.[illegible])}$$

$$\xi\sqrt{1-\xi^2} = \frac{1}{2\sqrt{10}}$$

$$\xi^2(1-\xi^2) = \frac{1}{40} \qquad \therefore \xi^4 - \xi^2 + \frac{1}{40} = 0 \qquad \xi^2 = \frac{1 \pm \sqrt{1-\frac{1}{10}}}{2}$$

$$= 0.974 \text{ or } 0.0256$$

$$\therefore \xi = \cancel{0.988} \text{ or } 0.16$$

왜) $\xi < 0.707$.

$$\therefore G_3(j\omega) = \frac{s^2 + 2\xi\omega_n s + \omega_n^2}{\omega_n^2} = \left(\frac{s}{\omega_n}\right)^2 + 2\xi\frac{s}{\omega_n} + 1 \quad \leftarrow \begin{array}{l}\omega_n = 2450 \approx \omega_r \\ \xi = 0.16\end{array}$$

$$= \left(\frac{s}{2450}\right)^2 + 0.32\left(\frac{s}{\omega_n}\right) + 1$$

⇒ Transfer fn.

$$T(s) = G_1(s)\cdot G_2(s)\cdot G_3(s)$$

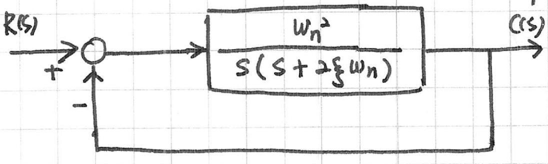
$$= \frac{\left(\frac{s}{2450}\right)^2 + 0.32\left(\frac{s}{2450}\right) + 1}{\left(1+\frac{s}{300}\right)\left(1+\frac{s}{20000}\right)} \quad //$$

8.5 Performance spec. in the freq. domain

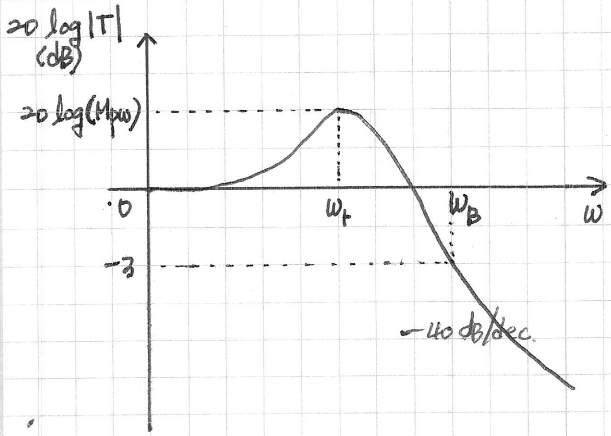
freq. response (Bode plot) $\longleftrightarrow$ transient resp. (time domain)

overshoot
t_r
t_s, ξ.

Example) 2nd order closed-loop system ~ Fig 8.24.



$$T(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}, \quad u \triangleq \frac{\omega}{\omega_n}$$



resonant freq. $\omega_r = \omega_n\sqrt{1-2\xi^2}$

max. mag. $M_{p\omega} = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

Fig 8.11.

※ Bandwidth $\omega_B \triangleq -3dB$ freq.

$0dB \to |T| = 1$

$$20\log|T| = -10\log\left((1-u^2)^2 + 4\xi^2u^2\right) = -3dB.$$

$|T| = \frac{1}{\sqrt{2}} = 0.707.$

$$(1-u^2)^2 + 4\xi^2u^2 = 10^{0.3} = 1.9955$$

⋮

$$\therefore u = \frac{\omega_B}{\omega_n} \cong -1.1961\xi + 1.8508$$

$0.3 \le \xi \le 0.8$. ~ Fig 8.26.

$\omega_B \propto \omega_n, 1/\xi$.

※ ω_B 증가 (ω_N 증가 or ξ 감소) $\longleftrightarrow$ T_r 감소 : $T_r \cong \dfrac{2.16\xi + 0.6}{\omega_n}$ Eq(5.11) Fig 5.9.

$M_{p\omega}$ 증가 (ξ 감소) $\longleftrightarrow$ overshoot 증가 : $P.O. = 100e^{-\xi\pi/\sqrt{1-\xi^2}}$ Eq(5.15) Fig C5.8).

ξ 일정, ω_N 증가 $\longleftrightarrow$ T_s 감소 : $T_s = 4\tau = \dfrac{4}{\xi\omega_N}$ Eq(5.13)

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※ Desirable freq. domain spec.

(1). relatively small $M_{p\omega}$, ex). $M_{p\omega} < 1.5$.

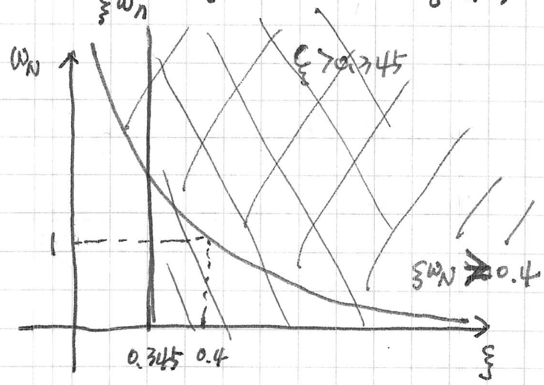
(2) 〃 large ω_B

Example) $M_{p\omega} < 1.5$, $T_s < 10$ sec. $\Rightarrow \xi, \omega_n = ?$

sol)

$M_{p\omega} < 1.5 \longrightarrow \xi > 0.345$ (from fig 8.11).

$T_s = \dfrac{4}{\xi \omega_n} < 10 \longrightarrow \xi\omega_n > 0.4$



eg). $\xi = 0.4$, $\omega_N = 2$

$\xi = 0.5$, $\omega_N = 2$

⋮

※ Steady-state error spec.

$$E(s) = R(s) - C(s) = [1 - T(s)] \cdot R(s).$$

(1). $R(s) = \dfrac{1}{s}$. (unit step)

$$e_{ss} = \lim_{s \to 0} sE(s) = 1 - T(0) = 1 - 1 = 0$$

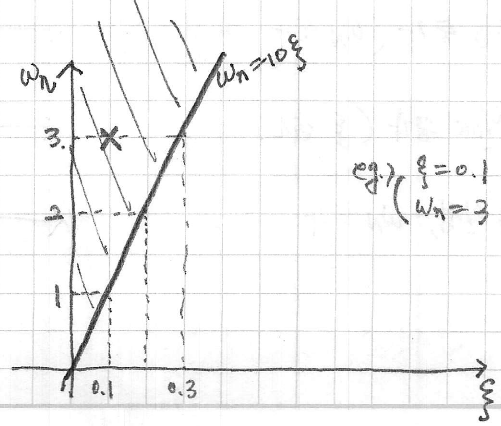
(2) $R(s) = \dfrac{1}{s^2}$ (unit ramp input).

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{1 - T(s)}{s} = \lim_{s \to 0} \left[\frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} \right] = \frac{2\xi}{\omega_n}$$

Example). $e_{ss} < 0.2$ for unit ramp input

sol). $\dfrac{2\xi}{\omega_n} < 0.2$. $\quad \dfrac{\xi}{\omega_n} < 0.1$

∴ $\omega_n > 10\xi$

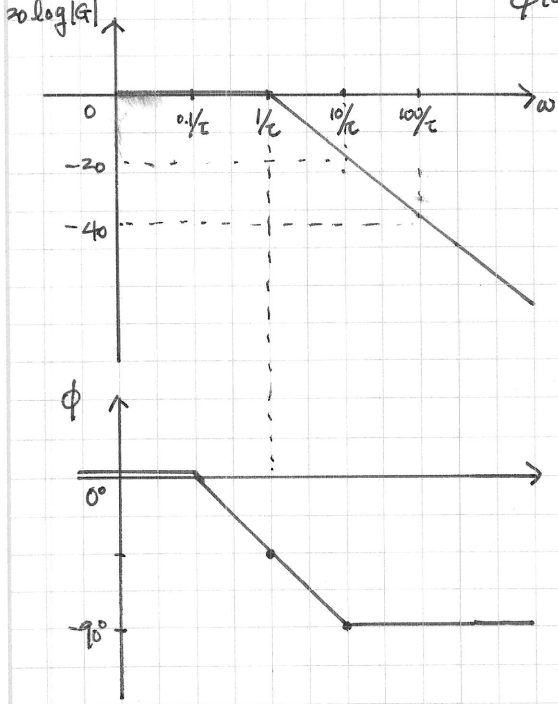


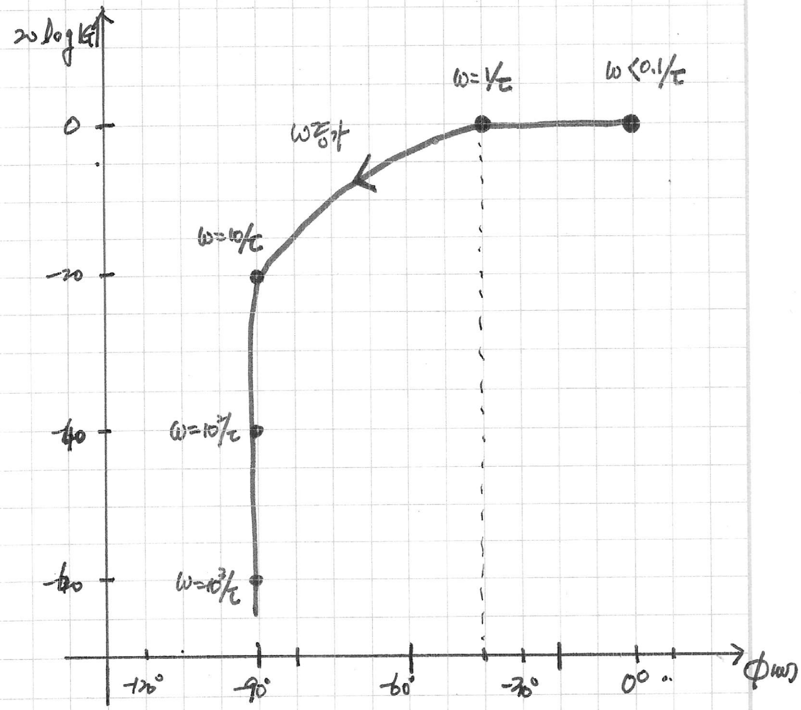
eg.) $\left(\begin{array}{l} \xi = 0.1 \\ \omega_n = 3 \end{array} \right)$

8.6 Log Mag. and Phase diagrams.

Ex) $G(j\omega) = \dfrac{1}{1+j\omega\tau}$, $20\log|G| = -10\log(1+\omega^2\tau^2)$

$\phi(\omega) = -\tan^{-1}(\omega\tau)$





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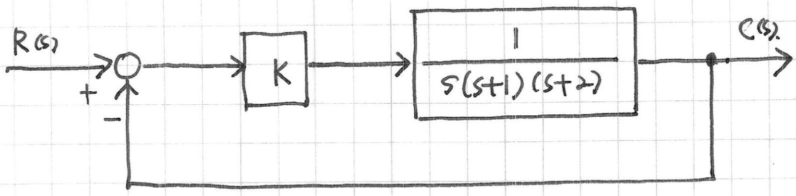
8.7. Design Example : Engraving machine control system



$$T(s) = \frac{K}{s^3+3s^2+2s+K}$$

To find K so that the unit step resp. is acceptable.

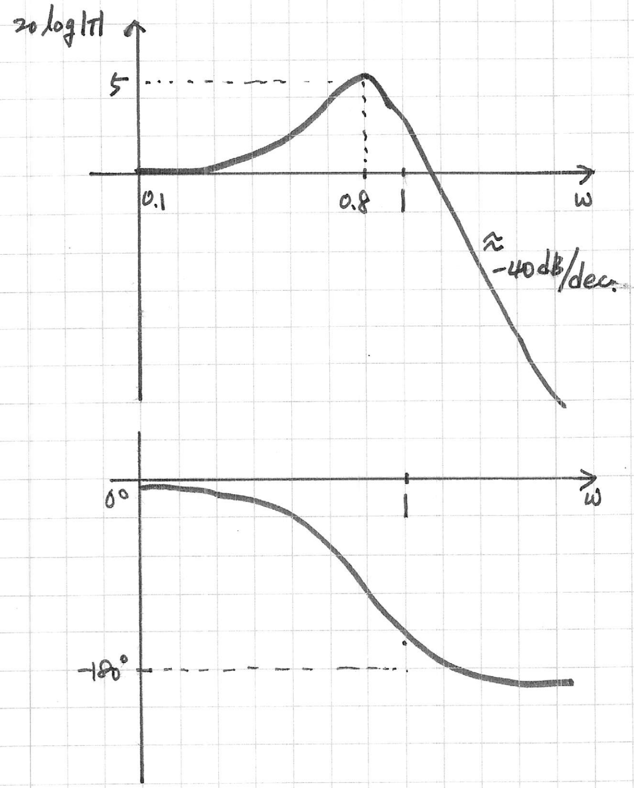
$\cancel{s_{1,2,3} = -2.5214}$
$\cancel{-0.3376 \pm j0.5623}$
dominant root

sol). If we select $K=2$.

$$T(j\omega) = \frac{2}{(2-3\omega^2)+j\omega(2-\omega^2)}$$

dominant pole.

$s_{1,2,3} = \cancel{-2.5214},\ -0.2393 \pm j0.8579$



approximate 2nd order system.

$$T(s) \cong \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$20\log|T| = 5\,dB$. at $\omega_r = 0.8$.

$\therefore |T| = 1.7783. \triangleq M_{p\omega}$

$\Rightarrow$ Fig 8.11 $\Rightarrow$ $\xi = 0.29$. 0.28.

$\dfrac{\omega_r}{\omega_n} = 0.91$

$\therefore \omega_n = \dfrac{0.8}{0.91} = 0.88.$

$$\Rightarrow T(s) = \frac{0.774}{s^2+0.51s+0.774}$$

$P.O. = 37\%$ for $\xi = 0.29$ from Fig 5.8.

$T_s = \dfrac{4}{\xi\omega_n} = 15.7$ sec.

*. If we require a system with lower overshoot,
we would reduce K to 1 and repeat the ~~proc~~ procedure.

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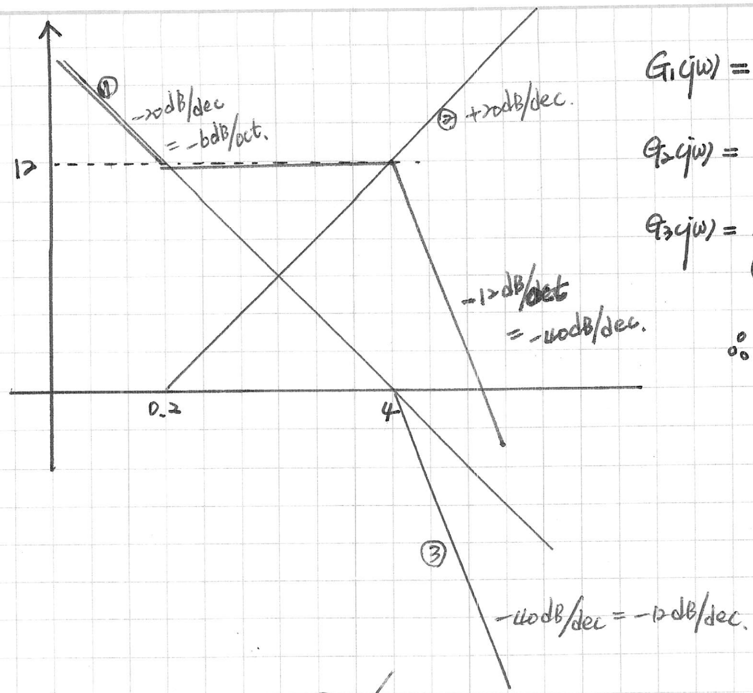
P8.6.



At $\omega = 0.2$

$$20 \log |G_1| = 12\,dB = 20 \log\left(\frac{k}{\omega}\right)\Big|_{\omega=0.2}$$

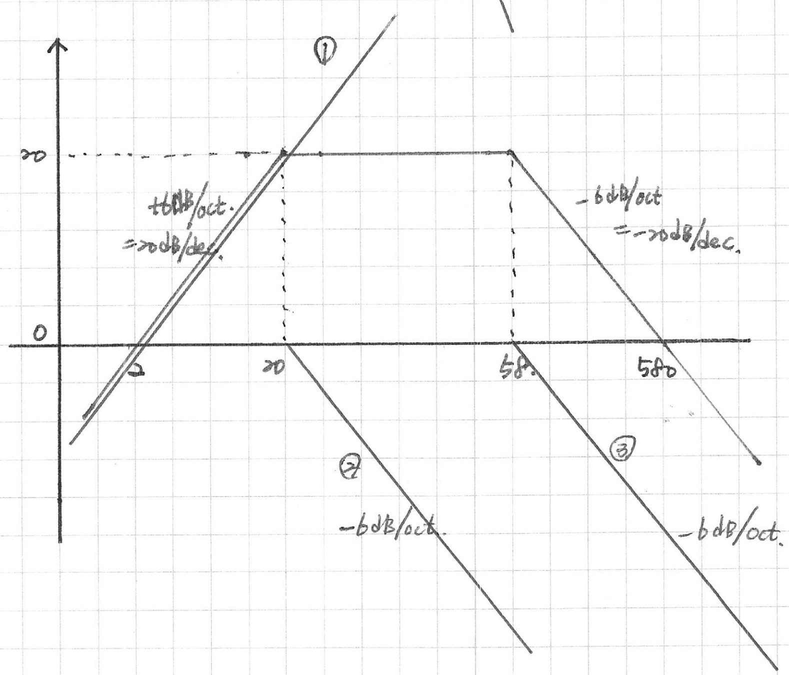
$$\therefore 5k = 10^{0.6} = 0.8$$

$$G_1(j\omega) = \frac{k}{j\omega}$$

$$G_2(j\omega) = 1 + j\omega/0.2$$

$$G_3(j\omega) = \frac{1}{(1 + j\omega/4)^2}$$

$$\therefore G(s) = G_1(s) \cdot G_2(s) \cdot G_3(s) = \frac{0.8\,(1 + s/0.2)}{s(1+s/4)^2}$$



$$G_1 = ks$$

$$G_2 = \frac{1}{1+s/20}$$

$$G_3 = \frac{1}{1+s/58}$$

$$20 \log |G_1|_{\omega=2} = 0\,dB$$

$$\Downarrow$$

$$20 \log(\omega k)\Big|_{\omega=2} = 0\,dB.$$

$$\therefore 2k = 0 \quad \therefore \underline{k = 0.5}$$

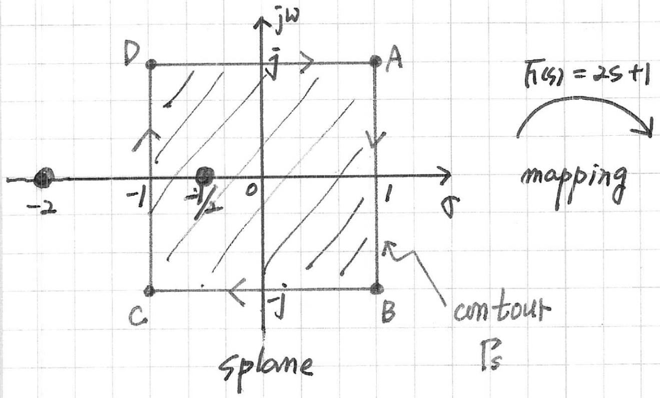
$$\therefore G(s) = \frac{0.5s}{(1+s/20)(1+s/58)}$$

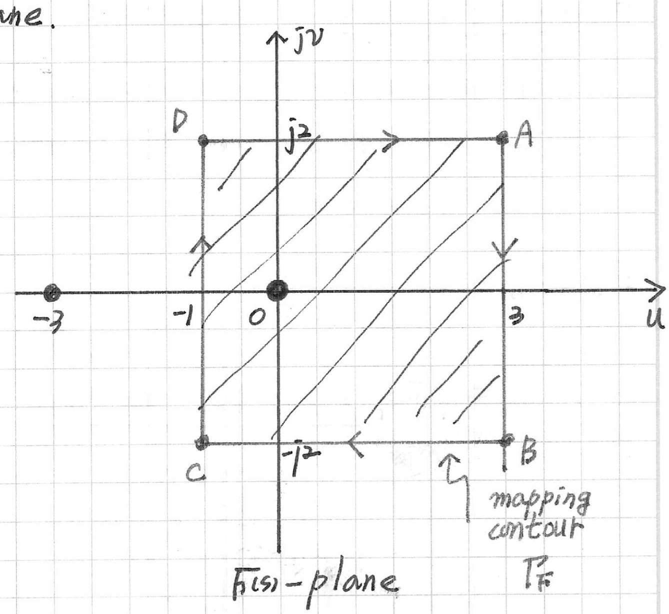
Chap. 9. Stability in the freq. domain

~ Nyquist stability criterion based on ~~the~~ Cauchy's thm.
(contour mapping)

9.2. Mapping contours in the s-plane.

Ex) $F(s) = 2s+1$





$$F(s) = 2s+1 = 2(\sigma + j\omega) + 1$$

$$= 2\sigma + 1 + j2\omega$$

$$\triangleq u + jv$$

$$\therefore \begin{pmatrix} u = 2\sigma + 1 \\ v = 2\omega \end{pmatrix}$$

ex). point A ($s = 1+j$) $\rightarrow F(s) = 2(1+j)+1 = 3+j2$ //

$s = -\frac{1}{2} \rightarrow F(s) = 2(-\frac{1}{2}) + 1 = 0$ //

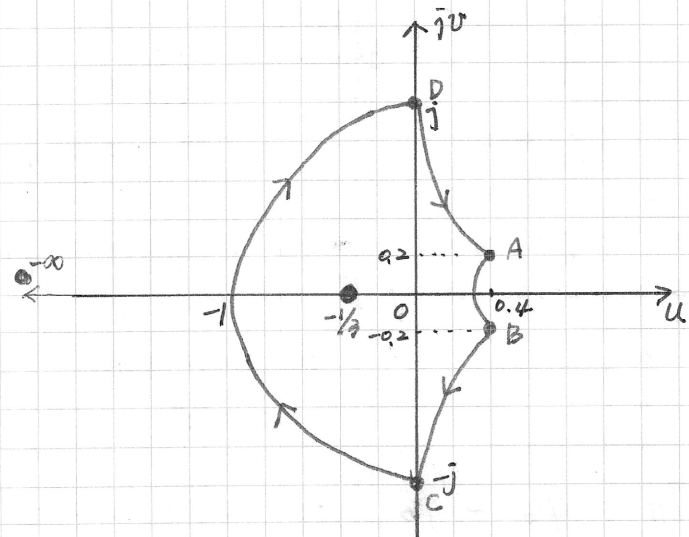
Ex) $F(s) = \dfrac{s}{s+2} = \dfrac{\sigma + j\omega}{\sigma + j\omega + 2}$

$$= \frac{\sigma^2 + 2\sigma + \omega^2 + j2}{(2+\sigma)^2 + \omega^2}$$

$$\triangleq u + jv$$

$$\therefore \quad u = \frac{\sigma^2 + 2\sigma + \omega^2}{(2+\sigma)^2 + \omega^2}$$

$$v = \frac{2}{(2+\sigma)^2 + \omega^2}$$



F(s) plane

Ex) point A, $s = 1+j \rightarrow F(s) = \dfrac{1+j}{3+j}$

$$= \frac{4+j2}{10}$$

$s = -1. \rightarrow F(s) = \dfrac{-1}{1} = -1.$

$$F(-2) = \frac{-2}{-2+2} = -\infty$$

* Enclosed area : contour area to the right of traversal of the contour

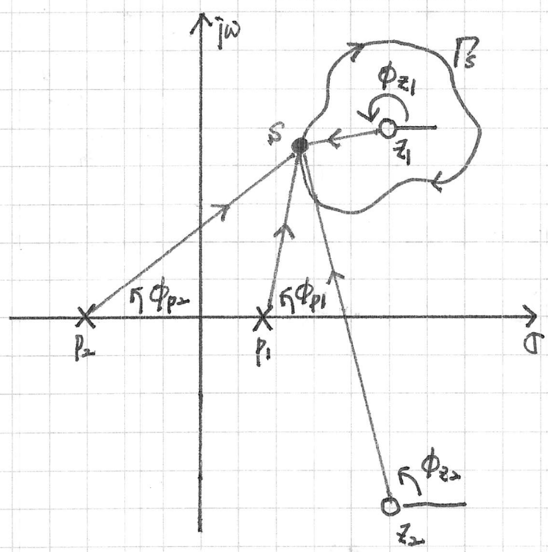
⇒ "clockwise and eyes right" rule

* Cauchy's thm ~ argument principle.

If a contour Γ_s enclose Z zeros and P poles of $F(s)$,
the mapping contour Γ_F enclose the origin of the $F(s)$-plane N times.
where $N = Z - P$.

Ex) $$F(s) = \frac{(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$$

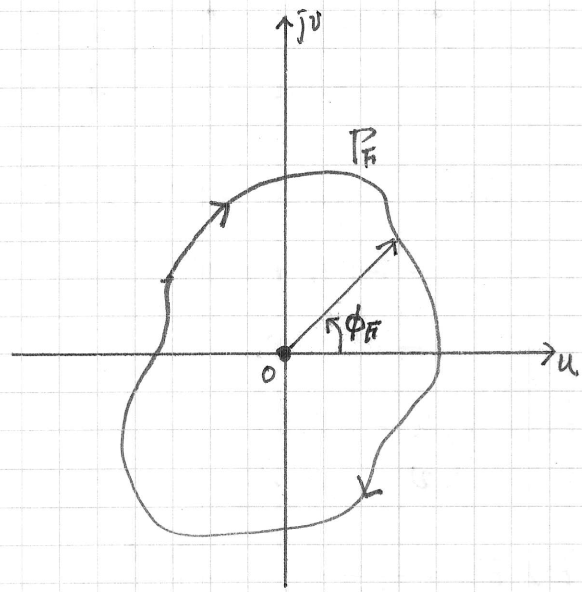
$$= |F(s)| \angle \overbrace{\phi_{z_1} + \phi_{z_2} - \phi_{p_1} - \phi_{p_2}}^{\phi_F}$$



s-plane

$\begin{cases} Z=1 \\ P=0 \end{cases}$

$\Rightarrow N = Z - P = 1$



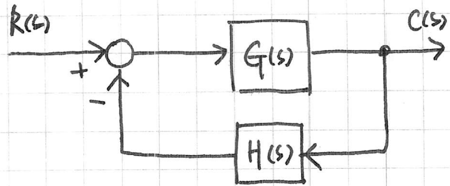
The net angle change for unenclosed poles and zeros (p_1, p_2, z_2)
as s traverses along Γ_s is $0°$.

" for enclosed " " (z_1)
" is $360°$

$\Rightarrow \phi_F = \phi_z - \phi_p$ OR $2\pi \cdot N = 2\pi \cdot Z + 2\pi \cdot P$ $\therefore$ $N = Z - P$.

where ϕ_F : net angle changes of $F(s)$.
ϕ_z : " of enclosed zeros.
ϕ_p : " of " poles.

9.3 Nyquist criterion



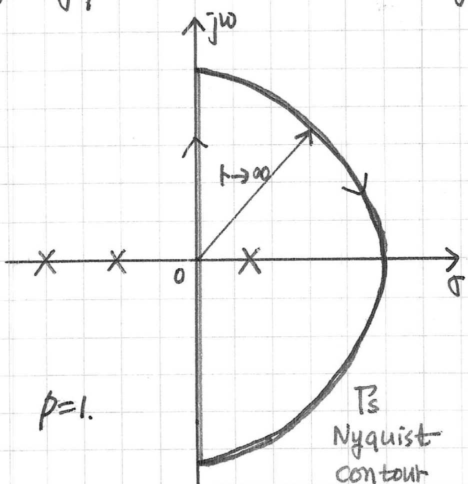
$$T(s) = \frac{G(s)}{1+GH(s)}$$

$F(s) \triangleq 1 + GH(s)$ ∶ $F(s)$의 zeros = system poles ($T(s)$의 pole)

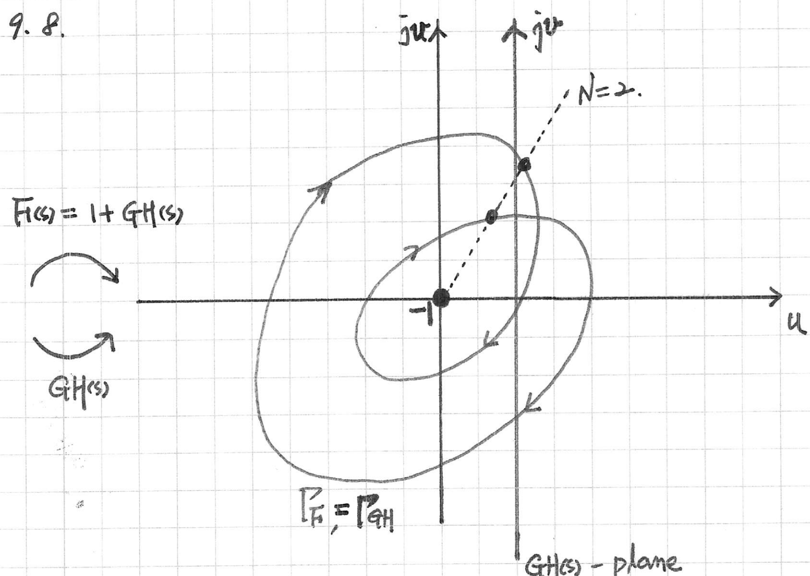
〃 poles = $GH(s)$의 poles

(note) $F(s)$의 zeros ≠ $GH(s)$의 zeros.

※ Nyquist contour ~ Fig 9.8.



p=1.



× ∶ $F(s)$의 poles = $GH(s)$의 poles.

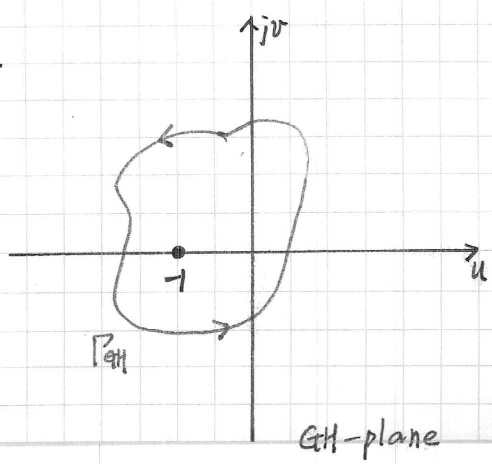
∴ $N = Z - P$ $\quad 2 = Z - 1$ $\quad$ ∴ $Z = 3$. (# of zeros of $F(s)$ in the rhp)

∴ unstable.

# of clockwise encirclements of the origin in the $F(s)$-plane.

= 〃 〃 of the $(-1,0)$ point in the $GH(s)$-plane

Ex).



$N = -1$ ∶ counterclockwise encirclement.

∴ $N = Z - P$

$Z = N + P = -1 + 1 = 0$.

∴ stable.

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※ Nyquist stability criterion ($Z=0$)

(1). When $P=0$, P_{GH} does not encircle the $(-1,0)$ point.

(2) When $P\neq 0$, # of counterclockwise encirclements of the $(-1,0)$ point $= P$.

proof).

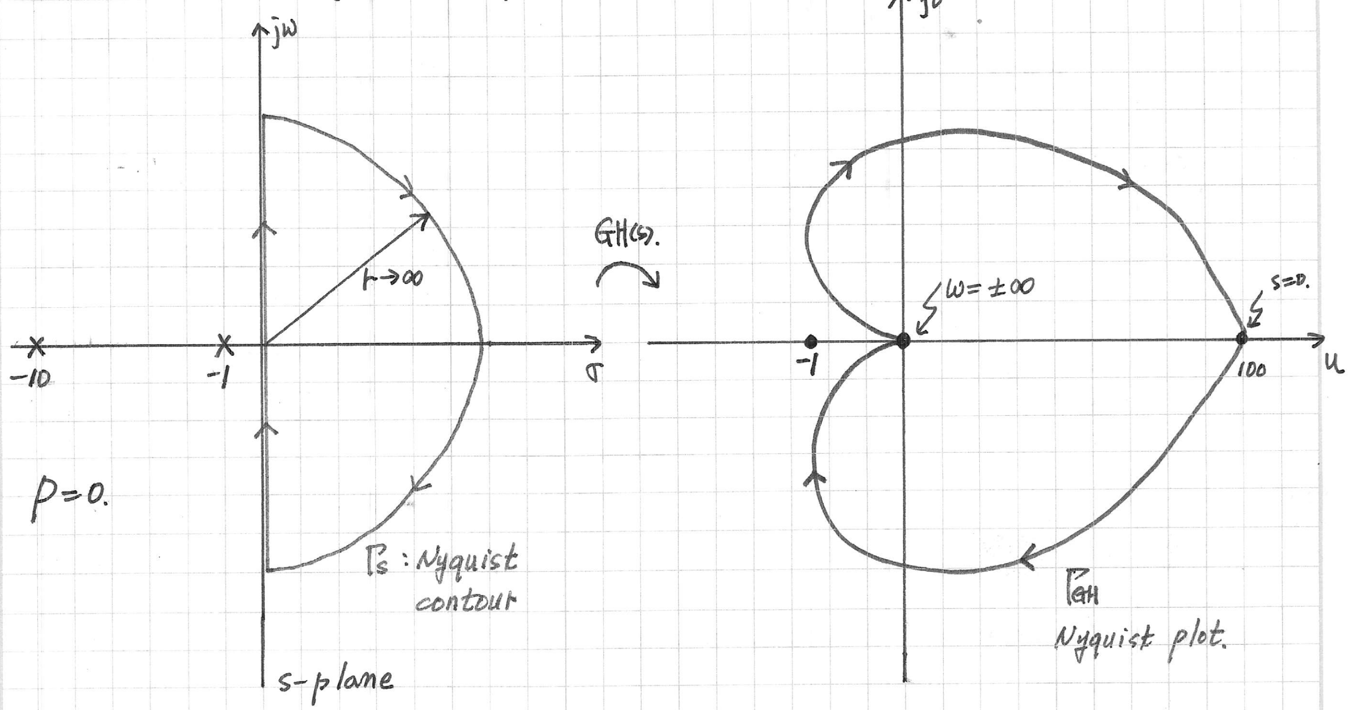
$$N = Z - P$$

(1). $P=0$, $N=Z$ ∴ if $N=0$, then $Z=0 \Rightarrow$ stable.

(2) $P\neq 0$, $Z=N+P$ ∴ if $N=-P$, then $Z=0 \Rightarrow$ stable.

Ex 9.1) System with 2 real poles.

$$GH(s) = \frac{1000}{(s+1)(s+10)} \quad \leftarrow \text{ char. eq } \; 1+GH(s)=0.$$

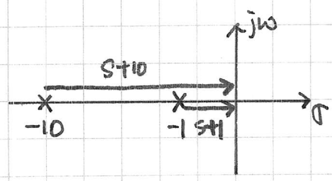


$p=0.$

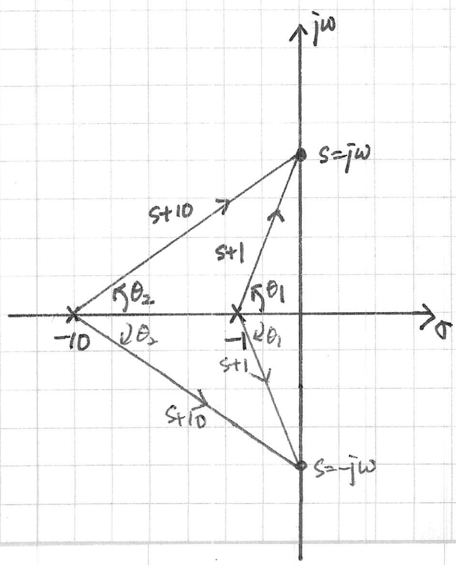
$$|GH(s)| = \frac{1000}{|s+1|\cdot|s+10|}, \qquad \angle GH(s) = -\angle s+1 - \angle s+10 = -(\theta_1+\theta_2)$$

(a) Origin $(s=0)$

$$\left(\begin{array}{l} |GH| = \dfrac{1000}{1\cdot 10} = 100 \\ \angle GH = -0^\circ \end{array}\right.$$



(b) $+j\omega$-axis ($s=j\omega$, $\omega=0^+, \sim \infty$)



$\omega = 0^+ \longrightarrow \infty$ (증가).

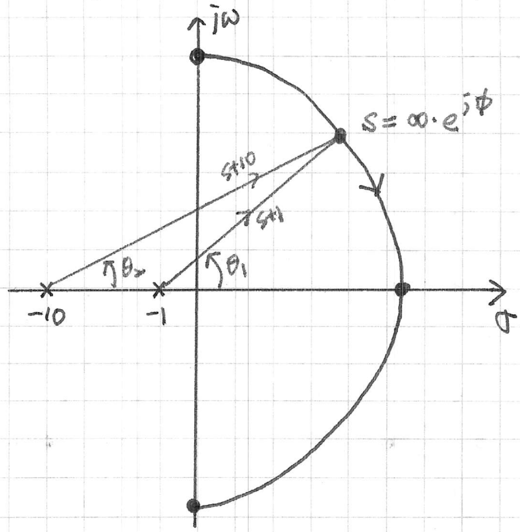
$$\left(\begin{array}{l} |GH| = 100 \longrightarrow 0 \text{ (감소)}. \\ \angle GH = 0^\circ \longrightarrow -180^\circ \end{array}\right.$$

(c) $-j\omega$ axis ($s=-j\omega$, $\omega=0^- \leftarrow -\infty$)

: symmetrical about the u-axis.

$$\left(\begin{array}{l} |GH| = 100 \leftarrow 0 \\ \angle GH = 0 \leftarrow 180^\circ \end{array}\right.$$

(d) infinite semi-circle ($s = re^{j\phi}$, $r \to \infty$, $\phi = +90° \to -90°$)



$$\left(\begin{aligned} |GH| &= \frac{1000}{\infty \cdot \infty} = 0 \\ \angle GH &= -180° \longrightarrow +180° \end{aligned} \right) \text{ 360° 회전.}$$

(another method)

$$\lim_{r\to\infty} GH(s)\Big|_{s=re^{j\phi}} = \lim_{r\to\infty} \frac{1000}{(\cancel{1}+re^{j\phi})(\cancel{10}+re^{j\phi})}$$

$$= \underbrace{\left(\lim_{r\to\infty} \frac{1000}{r^2}\right)}_{|GH|} \underbrace{e^{-j2\phi}}_{\angle GH}$$

$\therefore |GH| = 0$

$$\angle GH = -2\underbrace{\phi}_{\phi=90\to-90°} = -180 \to +180°$$

Nyquist plot does not encircle the (-1, 0) point.

$N = 0$

$\therefore Z = P + N = 0 + 0 = 0$ ∴ the system is "stable"

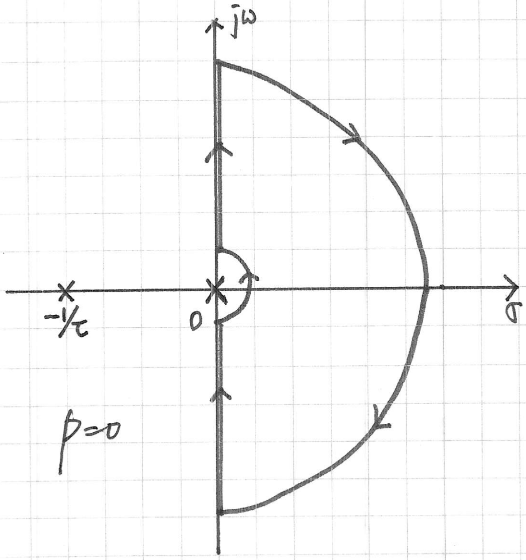
Ex 9.2) System with a pole at the origin

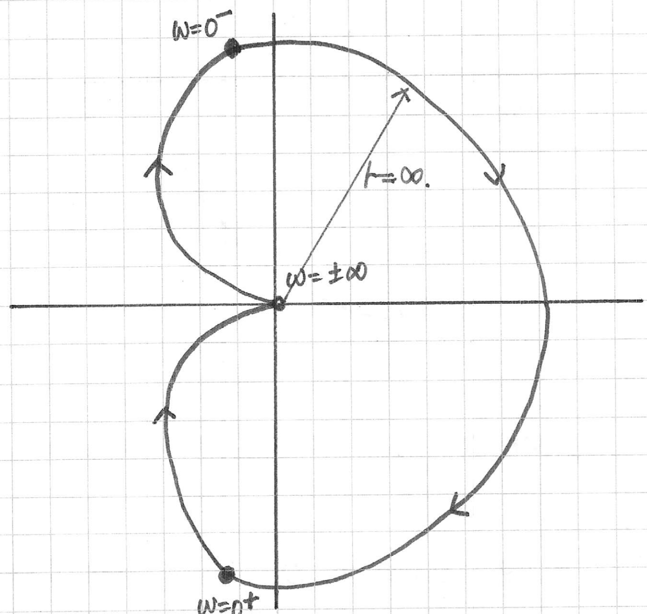
$$GH(s) = \frac{K}{s(\tau s+1)}, \quad \text{where } k>0.$$

$$\left(\begin{array}{l} |GH| = \dfrac{K/\tau}{|s|\cdot|s+1/\tau|} \\ \angle GH = -\angle s - \angle s+1/\tau = -(\theta_1+\theta_2) \end{array} \right.$$

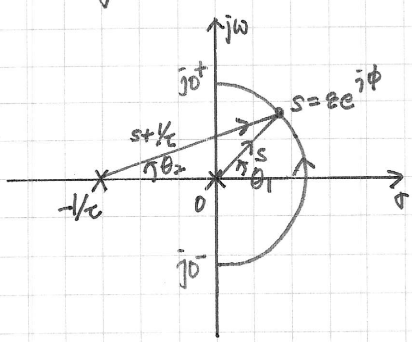
sol)

$$|GH(s)| = \frac{K/\tau}{|s|\cdot|\cancel{\tau}s+1/\cancel{\tau}|}, \quad \angle GH(s) = -\angle s - \angle \cancel{\tau}s+1/\tau = -(\theta_1+\theta_2)$$





(a) Origin ($s=\varepsilon e^{j\phi}$, $\varepsilon\to 0$, $\phi = -90^\circ \to +90^\circ$)



$$|GH(s)| = \frac{K/\tau}{0\cdot 1/\tau} = \infty.$$

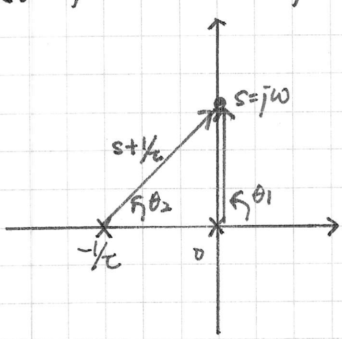
$$\underset{\omega=0^+}{\angle GH(s) = +90^\circ} \longrightarrow \underset{\omega=0^+}{-90^\circ} \quad \text{(over)}.$$

(another method)

$$\lim_{\varepsilon\to 0} GH(s)\Big|_{s=\varepsilon e^{j\phi}} = \lim_{\varepsilon\to 0} \frac{K}{\varepsilon e^{j\phi}\cdot(\cancel{\tau\cdot\varepsilon e^{j\phi}}+1)}$$

$$= \underbrace{\left(\lim_{\varepsilon\to 0}\frac{K}{\varepsilon}\right)}_{|GH|=\infty} \underbrace{e^{-j\phi}}_{\angle GH = +90 \to -90^\circ}.$$

(b) $+j\omega$-axis ($s=j\omega$, $\omega=0^+ \to \infty$)



$\omega = 0^+ \to \infty$

$$\left(\begin{array}{l} |GH| = \underset{\omega=0^+}{\infty} \longrightarrow \underset{\omega=\infty}{0} \\ \angle GH = -90^\circ \longrightarrow -180^\circ \end{array} \right.$$

(c) $-j\omega$ axis ($s=-j\omega$, $\omega = -\infty \to 0^-$)

: symmetrical about u-axis.

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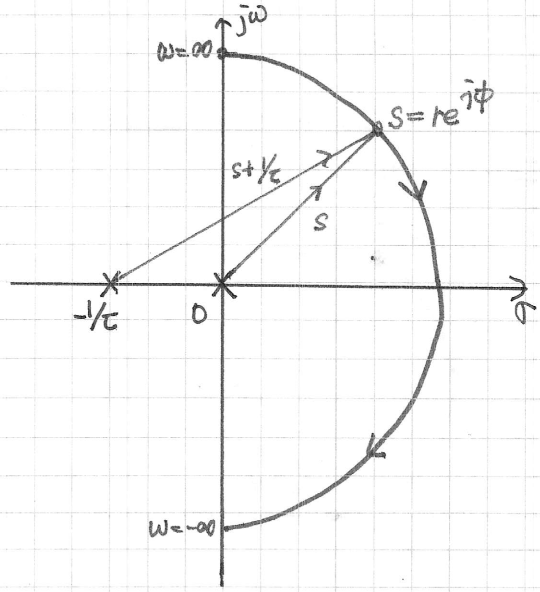
(d) infinite semi-circle ($s = re^{j\phi}$, $r \to \infty$, $\phi = +90° \to -90°$)



$$\left(\begin{aligned} |GH| &= \frac{K/\tau}{\infty \cdot \infty} = 0 \\ \angle GH &= -180° \to +180° \end{aligned} \right) \text{ 360° 회전.}$$

Nyquist plot does not encircle the $(-1, 0)$ point.

$N = 0$.

$\therefore Z = N + P = 0$ $\quad\therefore$ the system is "stable".

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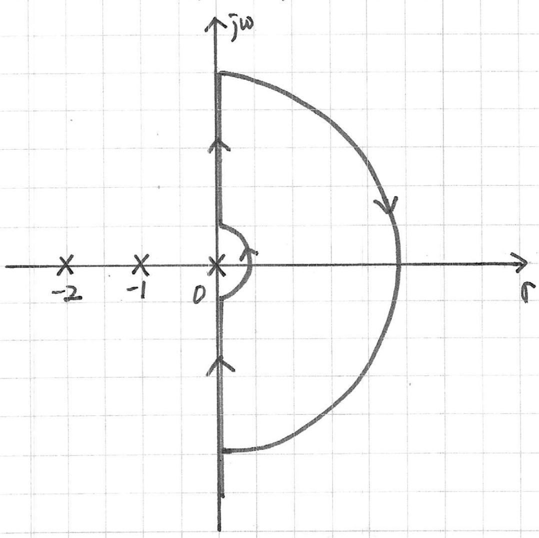
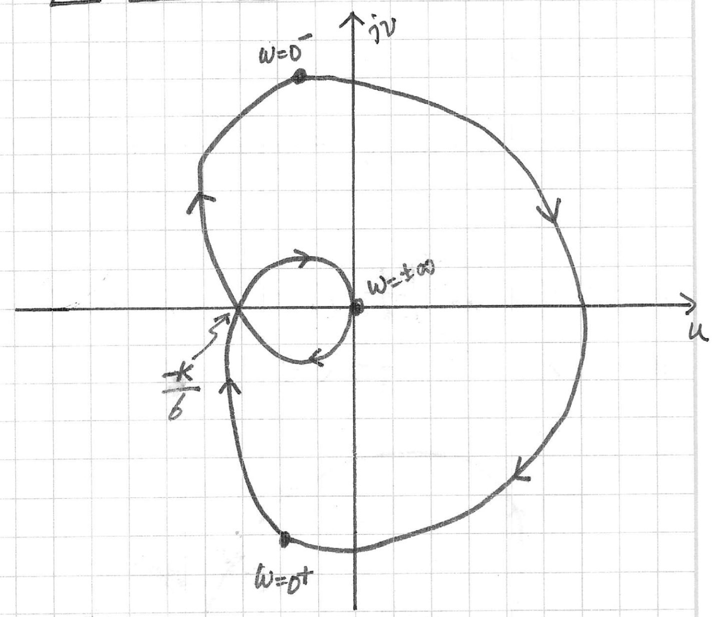
Ex 9.3) system with 3 poles.

$$GH(s) = \frac{K}{s(s+1)(s+2)}, \quad K > 0.$$

sol)

$$|GH| = \frac{K}{|s|\cdot|s+1|\cdot|s+2|}, \quad \underline{\angle GH} = -\angle s - \angle s+1 - \angle s+2 = -(\theta_1 + \theta_2 + \theta_3)$$

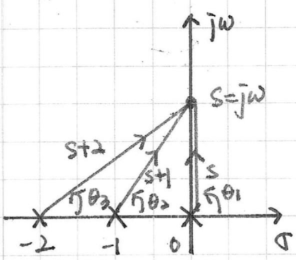




(a) origin ($s = \varepsilon e^{j\phi}$, $\varepsilon \to 0$. $\phi = -90^\circ \to +90^\circ$)

$$|GH| \cong \frac{K}{0\cdot 1\cdot 2} = \infty. \qquad \underline{\angle GH} = +90 \longrightarrow -90^\circ \text{ (over)}.$$

(b) $+j\omega$-axis ($s = j\omega$, $\omega = 0^+ \longrightarrow \infty$)



~~|GH| =~~

$\omega = 0^+ \longrightarrow \infty$

$$\left(\begin{array}{l} |GH| = \infty \longrightarrow 0 \\ \underline{\angle GH} = \underset{\omega=0^+}{-90^\circ} \longrightarrow \underset{\omega=\infty}{-270^\circ} \end{array} \right.$$

(c) $-j\omega$ axis ($s = -j\omega$, $\omega = -\infty \to 0^-$)

~ symmetrical.

(d) infinite semi-circle ($s = re^{j\phi}$, $r \to \infty$, $\phi = +90 \to -90^\circ$)

$$\left(\begin{array}{l} |GH| = \frac{K}{\infty\cdot\infty\cdot\infty} = 0. \\ \underline{\angle GH} = -270^\circ \longrightarrow +270^\circ \end{array} \right) \; 540^\circ \text{ 회전.}$$

※ Intersection with real axis [$Im(GH)=0$.]

$$GH(j\omega) = \frac{K}{j\omega(1+j\omega)(2+j\omega)} = \frac{jK(1-j\omega)(2-j\omega)}{-\omega(1+\omega^2)(4+\omega^2)}$$

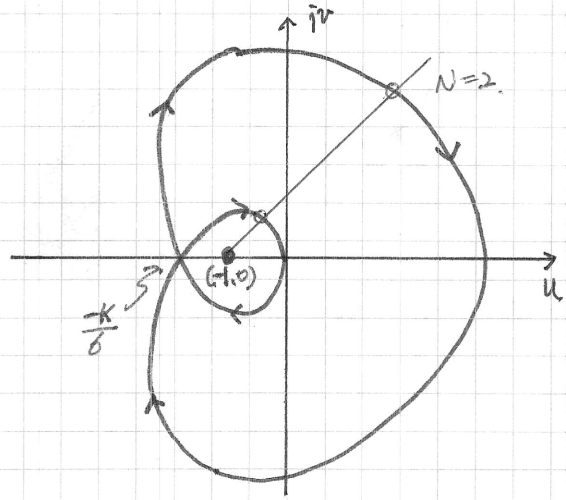
$$= \frac{3K\cancel{\omega}}{-\cancel{\omega}(1+\omega^2)(4+\omega^2)} + j\frac{K(2-\omega^2)}{-\omega(1+\omega^2)(4+\omega^2)}$$

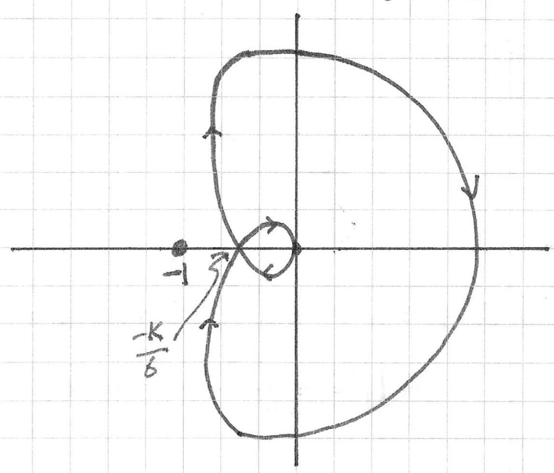
$Im\{GH\}=0 \Rightarrow \omega^2 = 2$,

$\therefore GH(j\omega)\Big|_{\omega^2=2} = \frac{3K}{-3\cdot 6} = -\frac{K}{6}$,

(1). $K<6$, $\frac{-K}{6} > -1$ $\therefore N=0$. $\therefore Z=N+P=0$ $\therefore$ stable.

(2) $K>6$ $\frac{-K}{6} < -1$ $\therefore$





$\therefore Z=N+P=2$.

"Unstable."

(3) $K=6$. $\frac{-K}{6}=-1$.

$\Rightarrow$ marginally stable.

cf) char. eq.

$s^3+3s^2+2s+K=0$.

(1) $k=6$ $s_{1,2,3} = -3,\ \pm j\sqrt{2}$ (임계 안정).

(2) $k=3$ " $= -2.6717,\ -0.1642 \pm j1.0469$

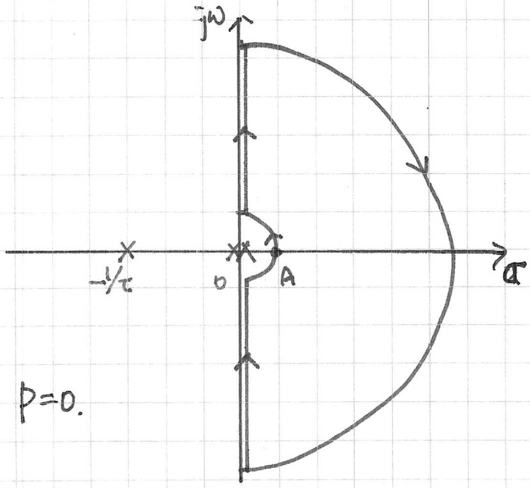
(3) $k=9$ " $= -3.24,\ 0.12 \pm j1.6623$

Ex 9.4) System with 2 poles at the origin

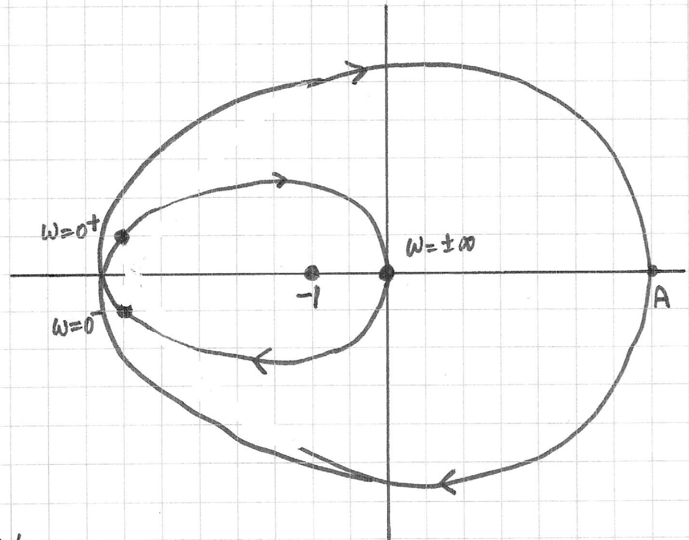
$$GH(s) = \frac{K}{s^2(\tau s+1)} = \frac{K/\tau}{s^2\cdot(s+1/\tau)}$$

sol)

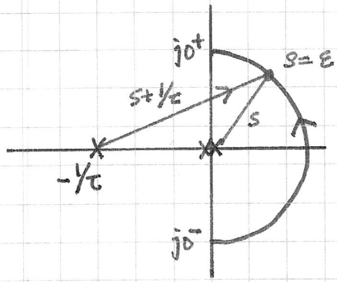
$$|GH| = \frac{K/\tau}{|s|^2\cdot|s+1/\tau|}, \quad \angle GH = -2\angle s - \angle s+1/\tau = -(2\theta_1+\theta_2)$$



P=0.



(a) small semicircular detour ($s=\varepsilon e^{j\phi}$, $\varepsilon\to 0$, $\phi=-90\to+90°$)



$$\left(\begin{array}{l} |GH| = \dfrac{K}{0^2\cdot(\frac{1}{\tau})} = \infty. \\ \angle GH = \underset{\omega=0^-}{+180} \longrightarrow \underset{\omega=0^+}{-180°} \end{array}\right.$$

cf. pt A, $\angle GH = 0°$

(b) large semi-circle. ($s=re^{j\phi}$, $r\to\infty$, $\phi=+90°\to-90°$)

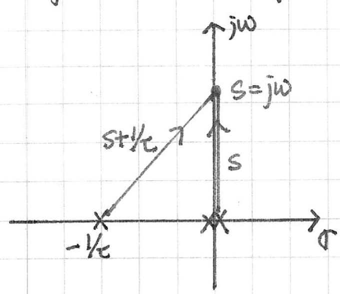
$$\left(\begin{array}{l} |GH| = \dfrac{K/\tau}{\infty^2\cdot\infty} = 0. \\ \angle GH = -270° \longrightarrow +270° \end{array}\right.$$ (540° 회전).

$N=2$

$\therefore Z=N+P=2$

⇒ "unstable"

(c) jω-axis ($s=j\omega$, $\omega=0^+\to\infty$)



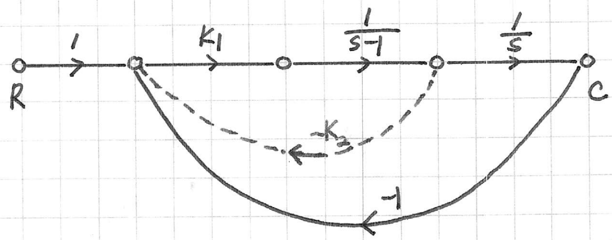
$\omega=0^+\to\infty$

$$\left(\begin{array}{l} |GH| = \infty \longrightarrow 0 \\ \angle GH = -180° \longrightarrow -270° \end{array}\right.$$

(d) -jω axis ~ symmetrical.

Ex 9.5) System with a pole in rhp

$$GH(s) = \frac{K_1(1+K_2 s)}{s(s-1)}$$



$$T(s) = \frac{P_1 \Delta_1}{\Delta} \triangleq \frac{C}{R}$$

where

$$\Delta = 1 - \left(\frac{-K_1 K_2}{s-1} + \frac{-K_1}{s(s-1)}\right)$$

$F(s) \;\; = 1 + \dfrac{K_1(1+K_2 s)}{s(s-1)}$

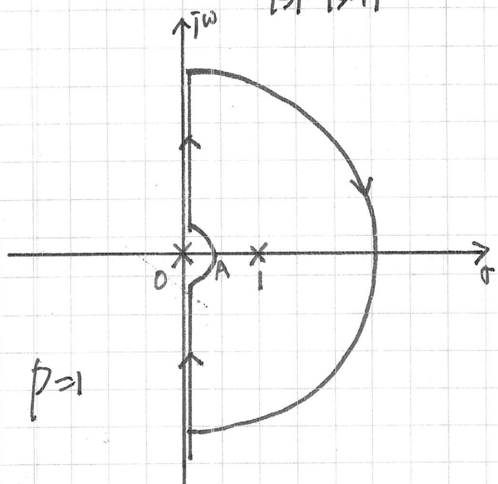
char. eq. $F(s) = 1 + GH(s)$

$$P_1 = \frac{K_1}{s(s-1)}, \quad \Delta_1 = 1.$$

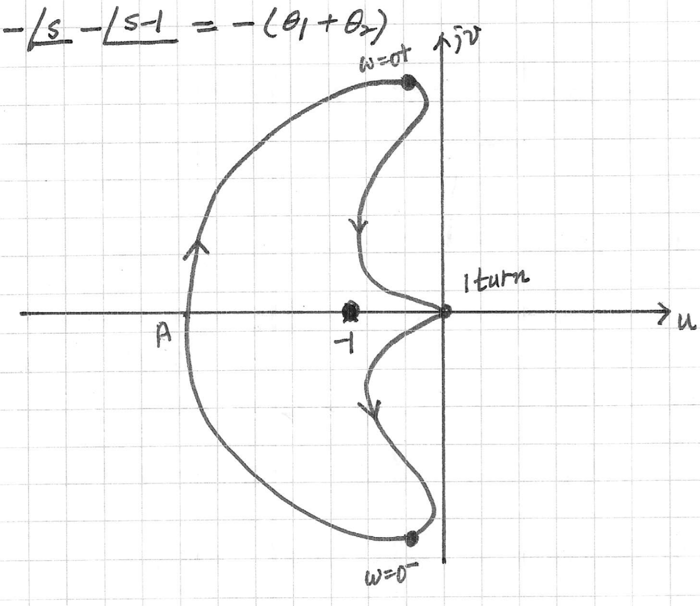
(1). $K_2 = 0.$

$$GH_1(s) = \frac{K_1}{s(s-1)}$$

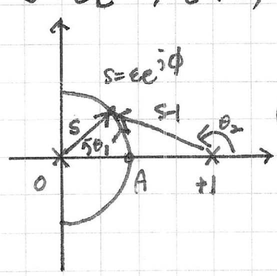
$$\Rightarrow |GH_1| = \frac{K_1}{|s|\cdot|s-1|}, \quad \angle GH_1 = -\angle s - \angle s-1 = -(\theta_1 + \theta_2)$$



$P = 1$



(a) $s = \varepsilon e^{j\phi}$, $\varepsilon \to 0$, $\phi = -90 \to 90°$

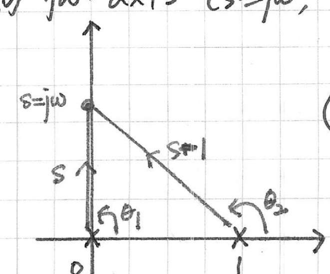


$$\begin{cases} |GH| = \frac{K_1}{0.1} = \infty \\ \angle GH = -90 \to -270° \\ \text{pt } A,\ \angle GH = -180° \end{cases}$$

$N = 1 \quad \therefore Z = N + P = 2$

$\therefore$ "unstable" for all K_1.

(b) $j\omega$-axis ($s = j\omega$, $\omega = 0^+ \to \infty$)



$$\begin{cases} |GH| = \infty \to 0 \\ \angle GH = -270° \to -180° \end{cases}$$

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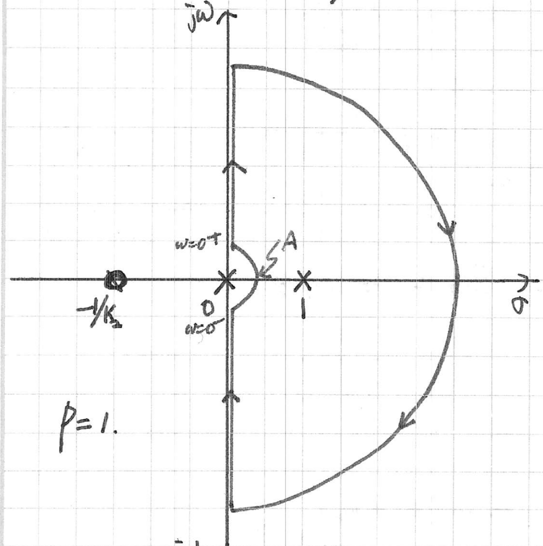
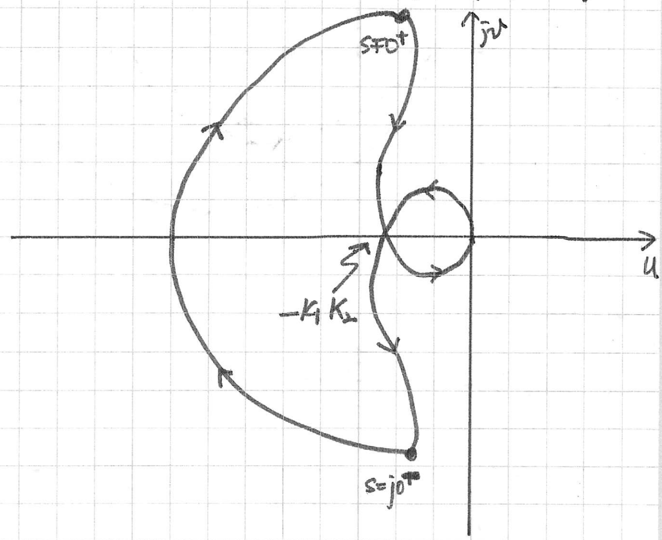
(2) $K_2 \neq 0$ (derivative f/b)

$$GH_2(s) = \frac{K_1(1+K_2 s)}{s(s-1)}, \quad |GH_2| = \frac{K_1 K_2 |s+1/K_2|}{|s|\cdot|s-1|}, \quad \angle GH_2 = \angle s+\tfrac{1}{K_2} - \angle s - \angle s-1$$
$$= \theta_z - (\theta_1 + \theta_2)$$



$p=1.$



(a) $s = \varepsilon e^{j\phi}$ ($\varepsilon \to 0$, $\phi = -90° \to 90°$)

$$|GH_2| = \frac{K_1K_2 \cdot \frac{1}{K_2}}{0 \cdot 1} = \infty, \quad \angle GH_2 = \underset{\omega=0^-}{-90°} \to \underset{\omega=0^+}{+90°}$$

(b) $s = j\omega$ ($\omega = 0^+ \to \infty$)

$$|GH_2| = \infty \to 0, \quad \angle GH_2 = \underset{\omega=0^+}{+90°} \longrightarrow \underset{\omega=\infty}{-90°} \quad (-90° \to 90°)$$

(c) $s = re^{j\phi}$ ($r \to \infty$, $\phi = +90° \to -90°$)

$$|GH_2| = 0, \quad \angle GH_2 = -90° \longrightarrow +90°.$$

※ Intersection with real axis.

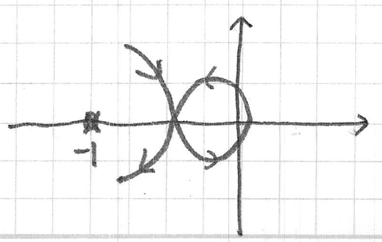
$$GH_2(j\omega) = \frac{K_1(1+K_2 \cdot j\omega)}{j\omega(j\omega-1)} = \frac{-K_1\omega(1+K_2) - jK_1(1-K_2\omega^2)}{\omega(\omega^2+1)}$$

$$\therefore \mathrm{Im}\{GH_2(j\omega)\} = 0 \Rightarrow 1 - K_2\omega^2 = 0 \quad \therefore \omega^2 = \frac{1}{K_2}$$

$$\therefore \mathrm{Re}\{GH_2(j\omega)\} = GH_2(j\omega)\Big|_{\omega^2=\frac{1}{K_2}} = \frac{-K_1(1+K_2)}{\frac{1}{K_2}+1} = \underline{\underline{-K_1K_2}}$$

① $K_1K_2 = 1 \longrightarrow$ marginally stable.

② $K_1K_2 < 1 \longrightarrow$ unstable.



$N=1$

$\therefore Z = N + P = 2$

③ $K_1K_2 > 1 \longrightarrow$ stable.

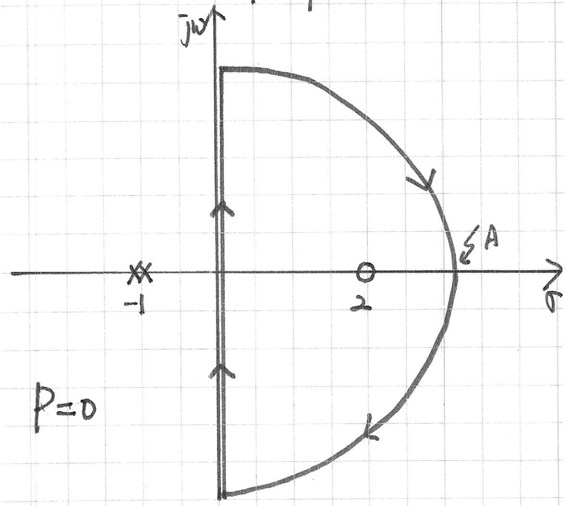
$N = -1$ (counterclockwise).

$\therefore Z = N + P = 0$

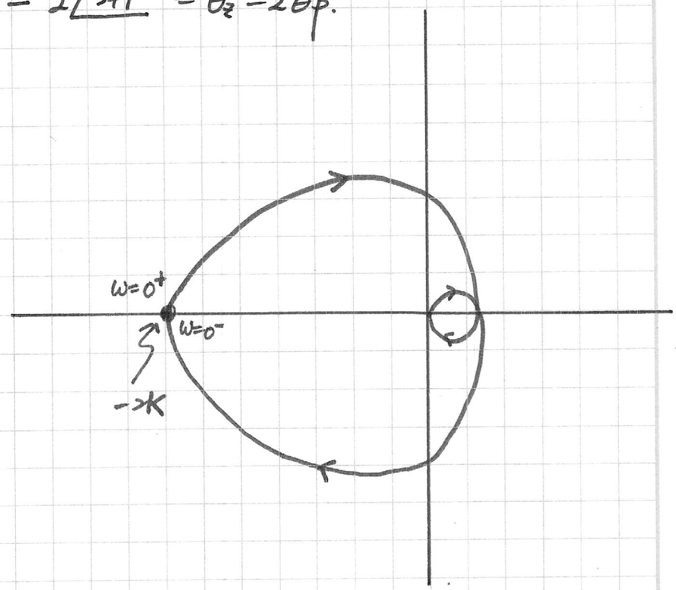
Ex 9.6) System with a zero in rhp.

$$GH(s) = \frac{K(s-2)}{(s+1)^2},$$

sol). $|GH| = \dfrac{K|s-2|}{|s+1|^2}$, $\angle GH = \angle s-2 - 2\angle s+1 = \theta_z - 2\theta_p$.



$P=0$



(1) $s=j\omega$, $\omega = 0^+ \longrightarrow \infty$.

$$\underset{\omega=0^+}{|GH| = 2K} \longrightarrow \underset{\omega=\infty}{0}, \quad \angle GH = \underset{\omega=0^+}{180^\circ} \longrightarrow 0 \longrightarrow \underset{\omega=\infty}{-90^\circ}$$

(2) $s = re^{j\phi}$, $r \to \infty$, $\phi = +90 \to -90^\circ$

$|GH| = 0$

$$\angle GH = \underset{\omega=+\infty}{-90^\circ} \longrightarrow \underset{A.}{0} \longrightarrow \underset{\omega=-\infty.}{+90^\circ}$$

If $K = \frac{1}{2} \rightarrow$ marginally stable.

$K < \frac{1}{2} \rightarrow N = 0 \quad \therefore Z = N + P = 0 \longrightarrow$ stable.

$K > \frac{1}{2} \rightarrow N = 1 \quad \therefore Z = 1 \longrightarrow$ unstable.

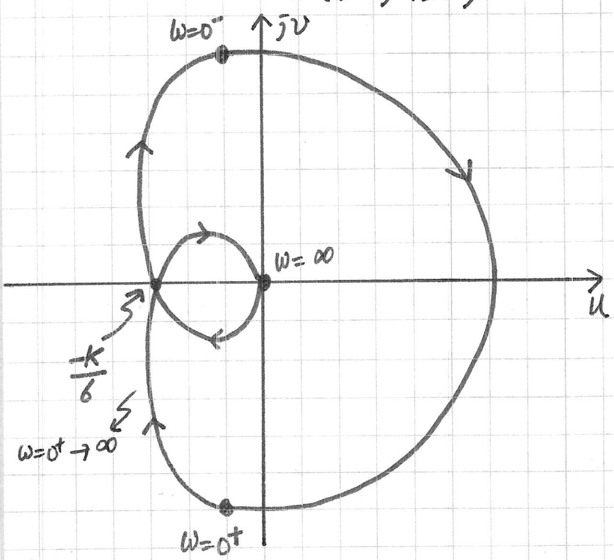
9.4 Relative stability and Nyquist criterion

$(-1,0)$ point on the polar plot of $GH(j\omega)$

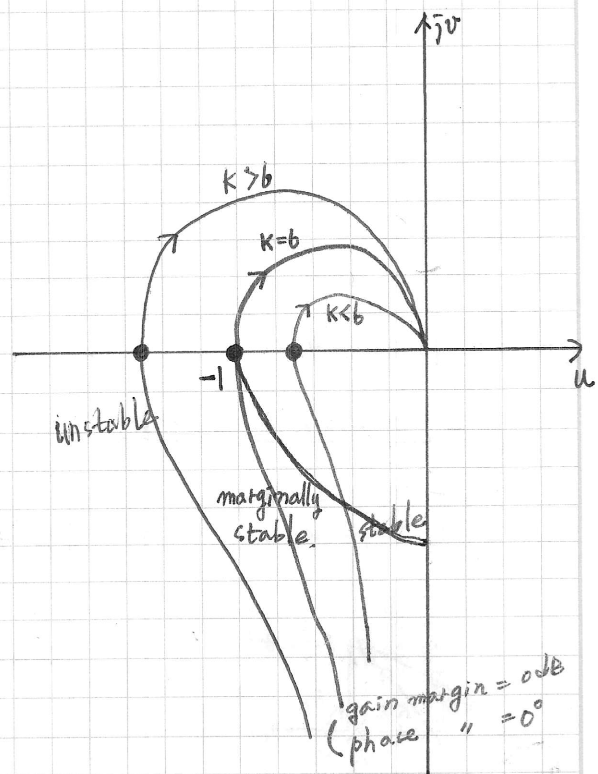
$\equiv$ $|GH(j\omega)| = 1$ and $\angle GH = -180°$ on the Bode plot.

Example) Return to Ex 9.3)

$$GH(s) = \frac{K}{s(s+1)(s+2)}$$



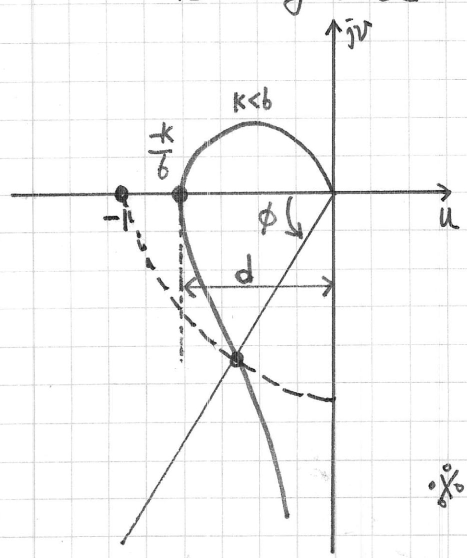
GH-plane



polar plot of $GH(j\omega)$

※ relative stability measure
(= gain margin)

: the margin between $(-1,0)$ point and $\frac{-K}{6}$



(1) gain margin $\triangleq \frac{1}{d}$ where $d = |GH(j\omega)|$ when $\angle GH = -180°$

$\Rightarrow$ logarithmic measure

$$20\log\left(\frac{1}{d}\right) = -20\log(d)$$

: 임계 안정 까지의 K의 증가에 대한 여유.

(2) phase margin $\triangleq \left| \angle GH \big|_{|GH|=1} - 180° \right|$

※ $K = 0^+ \to 6$ $\left(\begin{array}{l} \text{gain margin} = \infty\,\text{dB} \to 0\,\text{dB} \\ \text{phase} \;\; " \;\; = 90° \to 0° \end{array} \right)$

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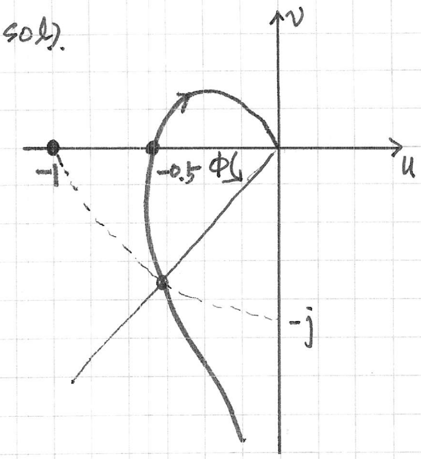
By ____ Date ____

Ex) $K=3$. i.e. $GH(s) = \dfrac{3}{s(s+1)(s+2)}$, gain margin, phase margin = ?

sol).



$d = 0.5$. gain margin $= \dfrac{1}{d} = 2$.

$20\log\left(\frac{1}{d}\right) = 20\log(2) \cong \underline{\underline{6dB.}}$

$|GH(j\omega)| = 1 \rightarrow \angle GH = ?$

$\therefore |j\omega|\cdot|1+j\omega|\cdot|2+j\omega| = 3$

$\sqrt{\omega^2(1+\omega^2)(4+\omega^2)} = 3$

$\therefore \omega^2(1+\omega^2)(4+\omega^2) = 9$

$\omega^6 + 5\omega^4 + 4\omega^2 - 9 = 0$

$\Rightarrow \omega = \pm 0.2504 \pm j1.7414, \pm 0.9693$

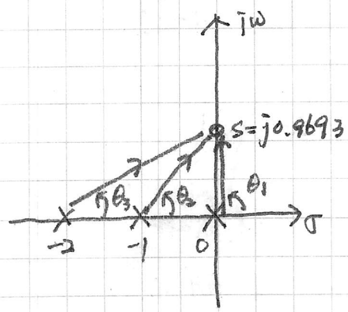
$\therefore \underline{\omega = 0.9693}$,,

$\angle GH(j\omega)\Big|_{\omega=0.9693}$

$= -\angle j\omega - \angle 1+j\omega - \angle 2+j\omega$

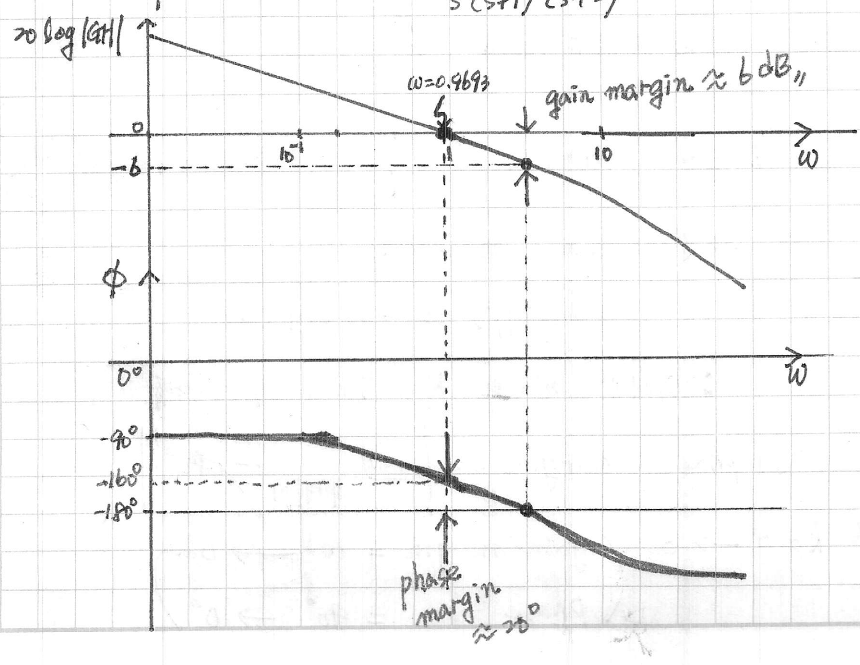
$= -(\theta_1 + \theta_2 + \theta_3)$

$= -(90^\circ + 44.1069^\circ + 25.8571^\circ) = -159.9640^\circ$

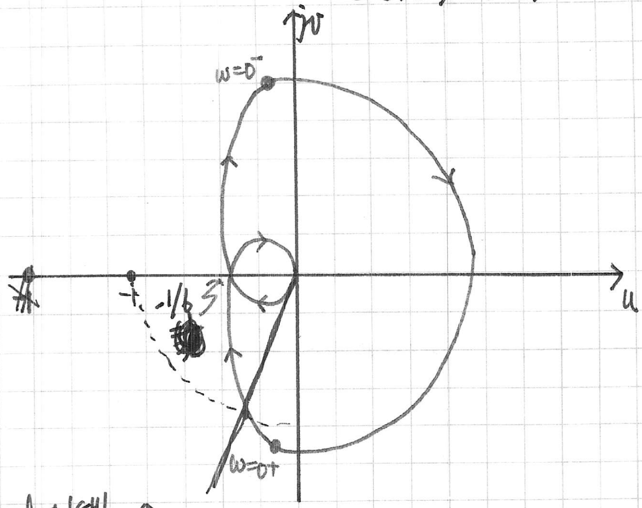


$\therefore$ phase margin $= 180^\circ - 159.9640^\circ = \underline{20.0360^\circ}$

※ Bode plot of $GH(s) = \dfrac{3}{s(s+1)(s+2)}$



Example) $GH(s) = \dfrac{5}{s(s+1)(s+5)}$ Eq. (9-50).

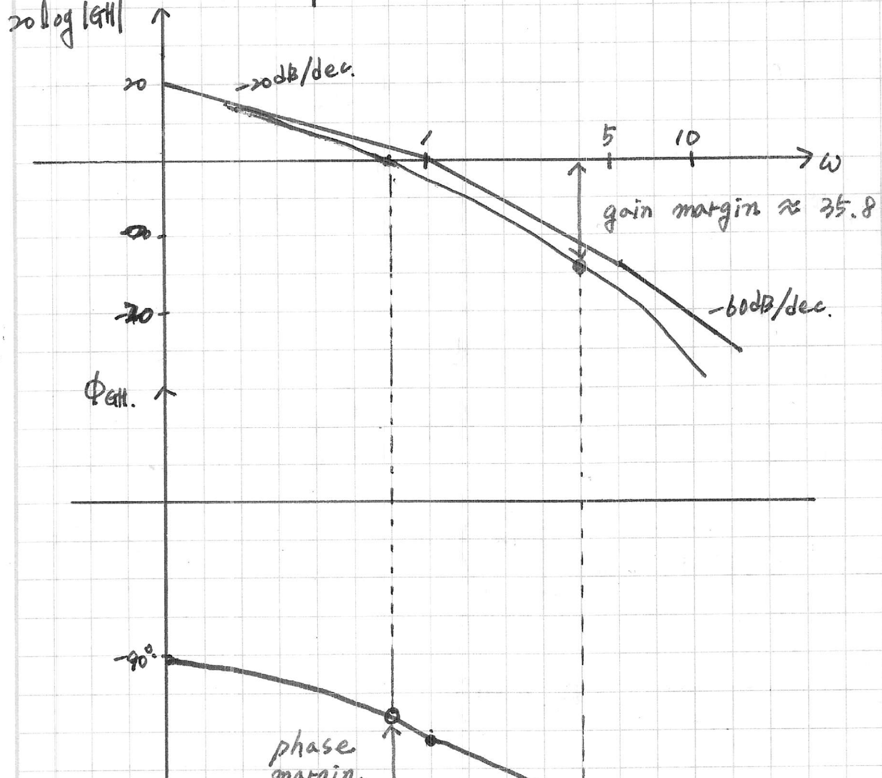


gain margin

$20 \log(\frac{1}{a}) = 20 \log(6) = 15.56$ dB.

phase margin

$\phi =$



※ Relation of ϕ_{pm} and ξ. for 2nd order system.

$$GH(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}, \qquad s^2 + 2\xi\omega_n s + \omega_n^2 = 0.$$

sol).

$$GH(j\omega) = \frac{\omega_n^2}{j\omega(j\omega + 2\xi\omega_n)}$$

$|GH(j\omega)|_{\omega=\omega_c} = 1$ where ω_c is cutoff freq. (crossover freq.)

$$= \frac{\omega_n^2}{\omega_c(\omega_c^2 + 4\xi^2\omega_n^2)^{1/2}}$$

$$\Rightarrow (\omega_c^2)^2 + 4\xi^2\omega_n^2(\omega_c^2) - \omega_n^4 = 0 \qquad \left(\frac{\omega_c^2}{\omega_n^2}\right)^2 + 4\xi^2\cancel{\omega_n^2}\left(\frac{\omega_c^2}{\omega_n^2}\right) - 1 = 0$$

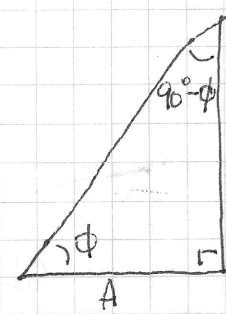
$$\therefore \frac{\omega_c^2}{\omega_n^2} = -2\xi^2 \underbrace{\pm}_{\text{trivial solution (음수)}} \sqrt{4\xi^4 + 1} \triangleq -2\xi^2 + \sqrt{4\xi^2+1} > 0.$$

phase margin for this system

$$\phi_{pm} = 180° + \angle GH(j\omega)\Big|_{\omega=\omega_c} = 180° - 90° - \tan^{-1}\frac{\omega_c}{2\xi\omega_n}$$

$$= 90° - \underbrace{\tan^{-1}\left(\frac{1}{2\xi}\left[\sqrt{4\xi^2+1} - 2\xi^2\right]^{1/2}\right)}_{\phi}$$

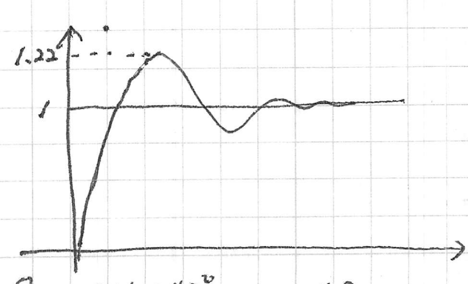
$$= \tan^{-1}\frac{2\xi}{\left[\sqrt{4\xi^2+1} - 2\xi^2\right]^{1/2}} \quad \text{(Eq. 9.57)}$$



$\tan\phi = \frac{B}{A}$

Approximation for $\xi \le 0.7$

$$\boxed{\xi \cong 0.01\,\phi_{pm}} \qquad \text{Eq. (9.58)}$$



Ex) previous example.

$$GH(s) = \frac{5}{s(s+1)(s+5)} \Rightarrow \phi_{pm} = 43° \qquad \therefore \xi = 0.01 \times 43° = 0.43$$

$\Rightarrow M_{pt} = 1.22$ from Fig 5.8.

9.6 System Bandwidth

the speed of unit step resp. $(t_r, t_p) \propto \omega_B$

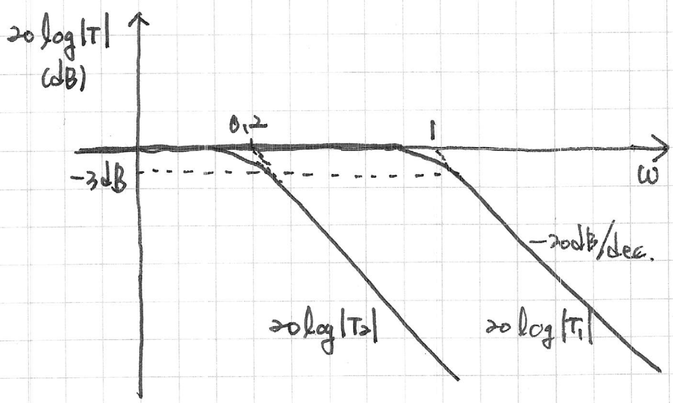
$T_s \propto \frac{1}{\omega_B}$

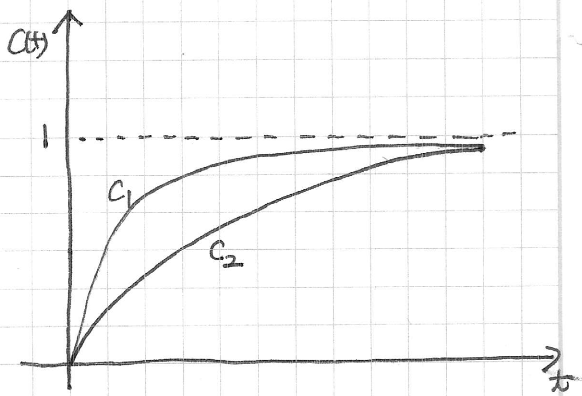
$$\Leftrightarrow \; t_r \cong \frac{2.16\xi + 0.6}{\omega_n}, \quad \frac{\omega_B}{\omega_n} \cong -1.1961\xi + 1.8508 \quad \text{(Fig. 8.26)}.$$

$$T_s = \frac{4}{\xi\omega_n}$$

Hence, "large Bandwidth"

Ex) $T_1(s) = \frac{1}{s+1}$, $T_2(s) = \frac{1}{5s+1}$





$\omega_{B1} = 1$, $\omega_{B2} = 0.2$

Ex) $T_1(s) = \frac{100}{s^2+10s+100}$, $T_2(s) = \frac{900}{s^2+30s+900}$: $\xi = 0.5$
$\omega_{n1} = 10$, $\omega_{n2} = 30$.

$\Rightarrow \omega_{B1} \cong 15$ $\omega_{B2} \cong 40$