Ch. 2. Mathematical models of systems

Differential eg. $\rightarrow$ Transfer fn
(Laplace transform)

2.2 Differential eg. of physical systems.

: by utilizing the physical laws of the process

$\begin{cases} \text{Newton's law for mechanical systems} \\ \text{Kirchhoff's law for electrical systems} \end{cases}$

Ex) Mass-Spring-Damping system : Fig 2.2

$$M\frac{d^2y(t)}{dt^2} + f\frac{dy(t)}{dt} + Ky(t) = r(t) \quad (2.1)$$

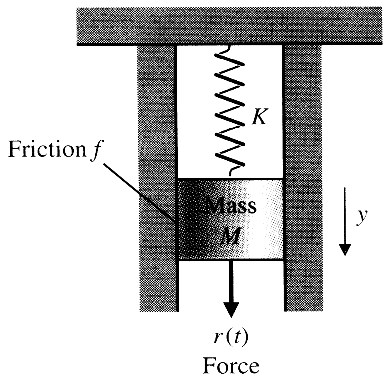
using Newton's 2 law of motion.

(2nd order linear constant coefficient diff. eq)

where $f\frac{dy(t)}{dt}$: friction (viscous damper)

$Ky(t)$: 복원력

Given $y(0)$, find $y(t)=?$

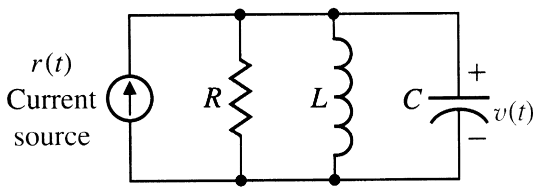


Ex) RLC circuit : Fig 2.3

$$\frac{v(t)}{R}+C\frac{dv(t)}{dt}+\frac{1}{L}\int_{-\infty}^{t} v(t)dt=r(t)$$

using Kirchhoff's current law

Given $v(0)$, find $v(t)=?$



※ Similarity between the diff. egs for the mechanical and electrical systems.

$v(t) \triangleq \frac{dy(t)}{dt}$ in (2.1) : velocity

$$\Rightarrow M\frac{dv(t)}{dt} + fv(t) + K\int_{\infty}^{t} v(t)dt = r(t)$$

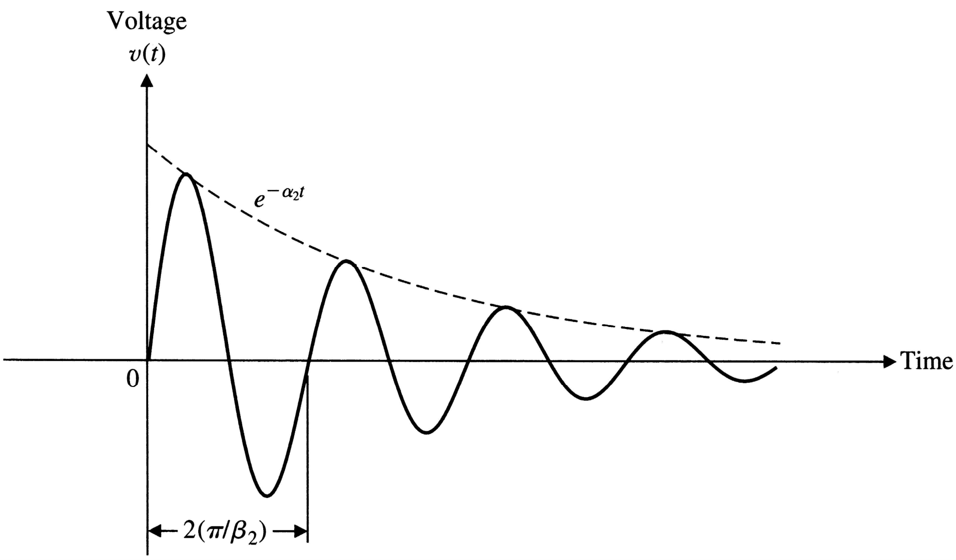
$f \leftrightarrow 1/R, \quad M \leftrightarrow C, \quad K \leftrightarrow 1/L$

$\therefore$ equivalent system (analogous system)

where velocity $\frac{dy(t)}{dt}$ and voltage $v(t)$ are equivalent(analogous) variables.

Generally, underdamped system

$v(t) = K_2 e^{-\alpha_2 t} \cos(\beta_2 t + \theta_2)$



2.3 Linear approximations of physical systems.

: Taylor's series expansion

<linear system : input x(t), output y(t) >

① superposition principle.

$x_1(t) \rightarrow y_1(t)$

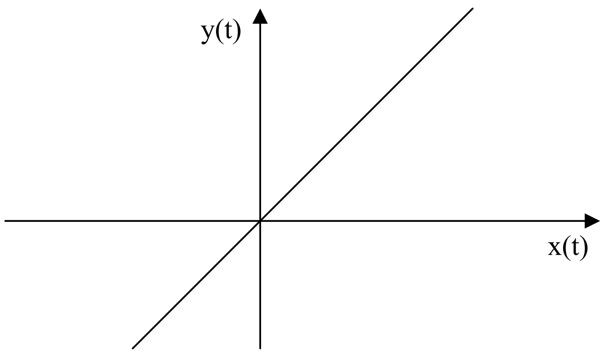
$x_2(t) \rightarrow y_2(t)$

$\Rightarrow x_1(t) \pm x_2(t) \rightarrow y_1(t) \pm y_2(t)$

② homogeneous principle

$x(t) \rightarrow y(t)$

$\Rightarrow \alpha\ x(t) \rightarrow \alpha\ y(t)$



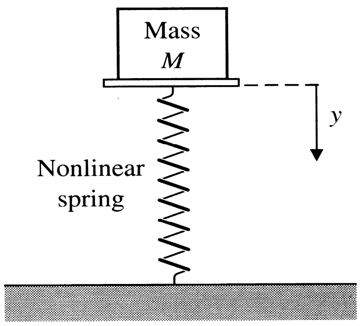
※ Taylor's series about x_0(operating point)

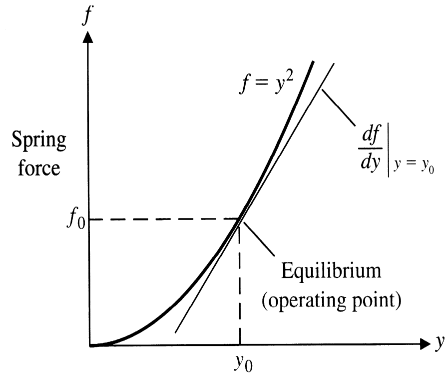
$y = g(x)$

$= g(x_0) + \frac{dg}{dx} \Big|_{x=x_0} (x - x_0) + \frac{d^2 g}{dx^2} \Big|_{x=x_0} \frac{(x-x_0)^2}{2!} + \cdots$

Ex) Mass sitting on a nonlinear spring : Fig 2.5

$f = y^2$

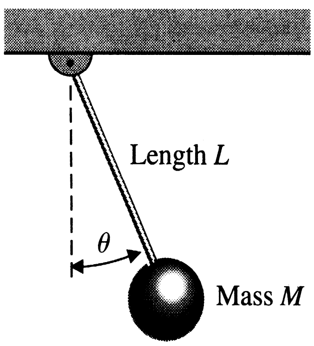




linear approximation about y_0 (equilibrium point)

$$f = y_0^2 + 2y \mid_{y=y_0} \cdot (y - y_0) + \cdots$$
$$\cong y_0^2 + 2y_0 y - 2y_0^2$$
$$\cong 2y_0 y - y_0^2$$

Ex 2.1) Pendulum oscillator model



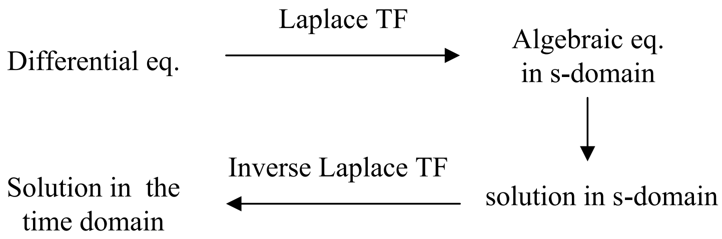
Torque

$$T = r \times F$$
$$= LMg\ \sin(180 - \alpha)$$
$$= Mg\ L\ \sin(\theta)$$
$$= f\ (\theta)$$

Linear approximation about $\theta = 0^\circ$

$$T = MgL\sin(0^\circ) + MgL\sin'(\theta) \mid_{\theta = 0^\circ} (\theta - 0^\circ) + HOT$$
$$\cong MgL\theta$$

2.4 The Laplace Transform



(1) definition

Laplace TF pair for a function of time. $f(t)$

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st}\, dt \equiv L\{f(t)\} \qquad f(t) = \frac{1}{2\pi j}\int_{r-j\theta}^{r+j\theta} F(s) e^{st} ds \equiv L^{-1}\{F(s)\}$$

※ Existence condition

$$\int_0^{\infty} |f(t)| e^{-\sigma t} dt < \infty$$

Example) $f(t) = u_s(t) = \begin{cases} 1, & t \geqq 0 \\ 0, & t < 0 \end{cases}$

sol) $F(s) = \int_0^{\infty} u_s(t) e^{-st} dt = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \frac{1}{s}$

Example) $f(t) = e^{-\alpha t},\ t \geq 0 = e^{-\alpha t} u_s(t)$

sol) $F(s) = \int_0^{\infty} e^{-\alpha t} e^{-st} dt = \int_0^{\infty} e^{-(s+\alpha)t} dt = \frac{1}{s+\alpha} e^{-(s+\alpha)t} \Big|_0^{\infty} = \frac{1}{s+\alpha}$

(2) Important theorems

Theorem 1. constant product

$kf(t) \leftrightarrow kF(s)$, where k is a constant

Proof) $\int_0^{\infty} kf(t) e^{-st} dt = k\int_0^{\infty} f(t) e^{-st} dt = kF(s).$

Theorem 2. sum and difference

$f_1(t) \pm f_2(t) \leftrightarrow F_1(s) \pm F_2(s)$

Proof) $\int_0^{\infty}[f_1(t)\pm f_2(t)]e^{-st}dt = \int_0^{\infty}\left[f_1(t)e^{-st}\pm f_2(t)e^{-st}\right]dt$

$$= \int_0^{\infty} f_1(t)e^{-st}dt \pm \int_0^{\infty} f_2(t)e^{-st}dt$$

$$= F_1(s) \pm F_2(s)$$

Linearity theorem

$$a\,f(t) \pm b\,g(t) \quad \leftrightarrow \quad a\,F(s) \pm b\,G(s)$$

Example) $f(t) = \cosh(at) = \dfrac{e^{at}+e^{-at}}{2}, \; F(s)=?$

$$F(s) = L\left\{\frac{e^{at}}{2} + \frac{e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}$$

$$= \frac{s}{s^2-a^2}$$

Theorem 3. differentiations

$$\frac{df(t)}{dt} \leftrightarrow sF(s) - f(0), \quad \text{where} \quad f(0) = \lim_{t\to 0} f(t)$$

Proof) $\int_a^b u'(t)v(t)dt = u(t)v(t)\Big|_a^b - \int_a^b u(t)v'(t)dt$

$$\int_0^{\infty} f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^{\infty} - \int_0^{\infty} f(t)(-s)e^{-st}dt$$

$$= f(\infty)e^{-\infty} - f(0)e^{-0} + s\int_0^{\infty} f(t)e^{-st}dt$$

$$= sF(s) - f(0)$$

$$\frac{d^2f(t)}{dt^2} \leftrightarrow s^2F(s) - s\,f(0) - f'(0)$$

Proof) $L\{f''(t)\} = L\{[f'(t)]'\}$, let $f'(t) = g(t)$

$= L\{g'(t)\}$ $\quad G(s) = s\,F(s) - f(0)$

$= s\,G(s) - g(0)$ where $g(0) = f'(0)$

$= s^2F(s) - s\,f(0) - f'(0)$

Generally,

$$\frac{d^n f(t)}{dt^n} \leftrightarrow s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

Theorem 4. integrations

$$\int_{-\infty}^{t} f(t)\,dt \;\leftrightarrow\; \frac{1}{s}F(s) + \frac{1}{s}f^{(-1)}(0), \qquad \text{where } f^{-1}(0) = \int_{-\infty}^{0} f(t)dt$$

$$\int\int_{-\infty}^{t} f(t)dt^2 \;\leftrightarrow\; \frac{1}{s^2}F(s) + \frac{1}{s^2}f^{(-1)}(0) + \frac{1}{s}f^{(-2)}(0)$$

$$\vdots$$

$$\int\cdots\int_{-\infty}^{t} f(t)dt^n \;\leftrightarrow\; \frac{1}{s^n}F(s) + \frac{1}{s^n}f^{(-1)}(0) + \frac{1}{s^{n-1}}f^{(-2)}(0) + \cdots + \frac{1}{s}f^{(-n)}(0)$$

Proof) $L\left\{\int_{-\infty}^{t} f(t)\,dt\right\} = \int_{0}^{\infty} e^{-st}\left[\int_{-\infty}^{t} f(t)dt\right] dt, \qquad \text{let } u = -\frac{1}{s}e^{-st}, \quad v' = f(t)$

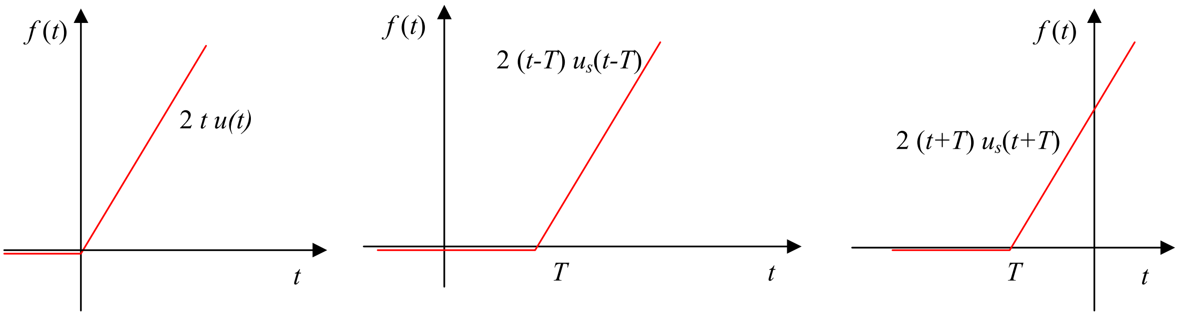
$$= -\frac{1}{s}e^{-st}\int_{-\infty}^{t} f(t)dt \,\Big|_{0}^{\infty} - \int_{0}^{\infty}\left(-\frac{1}{s}\right)e^{-st}f(t)\,dt$$

$$= \frac{1}{s}\int_{0}^{\infty} e^{-st} f(t)dt - \frac{1}{s}e^{-\infty}\int_{-\infty}^{\infty} f(t)dt + \frac{1}{s}e^{-0}\int_{-\infty}^{0} f(t)dt$$

$$= \frac{1}{s}F(s) + \frac{1}{s}\int_{-\infty}^{0} f(t)dt$$

Theorem 5. time shift

$$f(t \mp T)\,u_s(t-T) \;\leftrightarrow\; e^{\mp Ts}F(s)$$



Theorem 6. complex shift

$$e^{\mp at} f(t) \;\leftrightarrow\; F(s \pm a)$$

Theorem 7. Initial value theorem

$$f(0) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$

Theorem 8. Final value theorem (strictly stable system)

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Example) $F(s) = \dfrac{5}{s(s^2+s+2)}$ $\quad \lim_{t\to\infty} f(t) = ?$

sol) $\lim_{t\to\infty} f(t) = \lim_{s\to0} sF(s) = \lim_{s\to0} \dfrac{s\cdot 5}{s(s^2+s+2)} = \dfrac{5}{2}$

Example) $F(s) = \dfrac{w}{s^2+w^2}$

sol) $f(t) = \sin(wt)$ has two poles on the imaginary axis.(oscillation)

$\Rightarrow$ final value thm 적용 불가

Theorem 9. Real convolution

$$f_1(t) * f_2(t) \leftrightarrow F_1(s)\cdot F_2(s)$$

cf) complex convolution

$$f_1(t)\cdot f_2(t) \leftrightarrow F_1(s) * F_2(s)$$

<convolution>

$$\begin{aligned} f_1(t) * f_2(t) &= \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau \\ &= \int_0^t f_2(\tau) f_1(t-\tau)\, d\tau \end{aligned}$$

Theorem 10. differentiation in the s-domain

$$(-t)^n f(t) \leftrightarrow \frac{d^n F(s)}{ds^n}$$

$$\begin{aligned} \frac{dF(s)}{ds} &= \frac{d}{ds}\left[\int_0^\infty f(t) e^{-st}\, dt\right] \\ &= \int_0^\infty f(t) \frac{d}{ds}\left[e^{-st}\right] dt = \int_0^\infty (-t)\ f(t)\, e^{-st}\, dt \qquad \leftrightarrow \quad (-t) f(t) \end{aligned}$$

$$\frac{d^2F(s)}{ds^2} = \int_0^\infty f(t) \frac{d^2}{ds^2}\left[e^{-st}\right] dt = \int_0^\infty (-t)^2 f(t) e^{-st} dt \qquad \leftrightarrow (-t)^2 f(t)$$

$$\vdots$$

Theorem 11. integration in the s-domain

$$\frac{f(t)}{(-t)^n} \leftrightarrow \int_{s_1}^\infty \cdots \int_{s_n}^\infty F(s)\, d_{s_1} \cdots d_{s_n}$$

$$\frac{f(t)}{-t} \leftrightarrow \int_s^\infty F(s)\, ds \qquad \text{and} \qquad \frac{f(t)}{(-t)^2} \leftrightarrow \int \int_s^\infty F(s)\, d_{s_1} d_{s_2}$$

Example) $f(t) = t, \quad t \geq 0$

$$f(t) = t\,u_s(t)$$

$$(-t)u_s(t) \leftrightarrow \frac{d}{ds}\left[\frac{1}{s}\right] = \frac{-1}{s^2} \qquad \Leftarrow (-t)^n f(t) \leftrightarrow \frac{d^n}{ds^n}[F(s)]$$

$$(-1)\cdot(-t)u_s(t) \leftrightarrow (-1)\cdot\frac{-1}{s^2} \qquad \Leftarrow k\,f(t) \leftrightarrow k\,F(s)$$

hence, $t\,u_s(t) \leftrightarrow \dfrac{1}{s^2}$

Example) $f(t) = t^2, \quad t \geq 0$

$$(-t)^2 u_s(t) \leftrightarrow \frac{d^2}{ds^2}\left[\frac{1}{s}\right] = \frac{d}{ds}\left[\frac{-1}{s^2}\right] = \frac{2}{s^3}$$

Example) $f(t) = t^3, \quad t \geq 0$

$$(-t)^3 u_s(t) \leftrightarrow \frac{d^3}{ds^3}\left[\frac{1}{s}\right] = \frac{d}{ds}\left[\frac{2}{s^3}\right] = \frac{0-2\cdot 3s^2}{s^6} = \frac{-3!}{s^4}$$

hence, $t^3 u_s(t) \leftrightarrow \dfrac{3!}{s^4}$

Generally, $t^n \leftrightarrow \dfrac{n!}{s^{n+1}}$

Example) $f(t) = u_s(t-3) = \begin{cases} 1, & t \geq 3 \\ 0, & t < 3 \end{cases}$

$$u_s(t) \leftrightarrow \frac{1}{s}$$

$$u_s(t-3) \leftrightarrow \frac{e^{-3s}}{s} \qquad \Leftarrow f(t-T) \leftrightarrow e^{-Ts}F(s)$$

Example) $f(t) = (t-10)^2 u_s(t-10)$

$$t^2 u_s(t) \leftrightarrow \frac{2}{s^3}$$

$$(t-10)^2 u_s(t-10) \leftrightarrow \frac{2e^{-10s}}{s^3} \qquad \Leftarrow f(t-T) \leftrightarrow e^{-Ts}F(s)$$

Example) $f(t) = (t-10)^2 \qquad \Rightarrow f(t) = (t-10)^2 u_s(t) = \left(t^2 - 20t + 100\right) u_s(t)$

$$F(s) = \frac{2}{s^3} - \frac{20}{s^2} + \frac{100}{s} = \frac{2-20s+100s^2}{s^3}$$

Example) $f(t) \; = \; e^{3t}$

$$f(t) \; = \; e^{3t} = e^{3t} \cdot u_s(t)$$

$$u_s(t) \; \leftrightarrow \; \frac{1}{s} \; = \; F(s)$$

$$s^{3t} \; u_s(t) \; \leftrightarrow \; \frac{1}{s-3} \; = \; F(s-3) \qquad \Leftarrow \; e^{+\alpha t} f(t) \leftrightarrow \; F(s-\alpha)$$

Example) $f(t) \; = \; e^{3t} u_s(t-2)$

$$f(t) \; = \; e^{3t} u_s(t-2) \; = \; e^6 \, e^{3(t-2)} u_s(t-2)$$

$$s^{3t} \; u_s(t) \; \leftrightarrow \; \frac{1}{s-3}$$

$$s^{3(t-2)} \; u_s(t-2) \; \leftrightarrow \; \frac{e^{-2s}}{s-3}$$

$$s^6 \cdot s^{3(t-2)} \; u_s(t-2) \; \leftrightarrow \; \frac{e^6 \cdot e^{-2s}}{s-3} \; = \; \frac{e^{-2(s-3)}}{s-3}$$

Example) $f(t) \; = \; e^{3(t-2)} \; u_s(t)$

$$f(t) \; = \; e^{3(t-2)} \; u_s(t) = e^{-6} \, e^{3t} \; u_s(t) \quad \leftrightarrow \quad \frac{e^{-6}}{s-3}$$

<Laplace TF for sinusoidal function>

$$e^{jwt} \; = \; \cos(wt) + j \; \sin(wt)$$

$$e^{jwt} \; \leftrightarrow \; \frac{1}{s-jw} \cdot \frac{s+jw}{s+jw} = \frac{s+jw}{s^2+w^2} = \frac{s}{s^2+w^2} + j\frac{w}{s^2+w^2}$$

hence, $\cos(wt) + j \; \sin(wt) \; \leftrightarrow \; \dfrac{s}{s^2+w^2} + j\dfrac{w}{s^2+w^2}$

compare) $f_1(t) + j\,f_2(t) \; \leftrightarrow \; F_1(s) + j\,F_2(s)$

$$\cos(wt) \; \leftrightarrow \; \frac{s}{s^2+w^2}, \qquad \sin(wt) \; \leftrightarrow \; \frac{w}{s^2+w^2}$$

Example) $f(t) \; = \; e^{-at} \cos(wt)$

$$\cos(wt) \leftrightarrow \frac{s}{s^2+w^2}$$

$$e^{-at} \cos(wt) \leftrightarrow \frac{s+a}{(s+a)^2+w^2} \quad \Leftarrow \quad e^{-at} f(t) \; \leftrightarrow \; F(s+a)$$

Example) $f(t) = te^{2t}u_s(t)$

$$u_s(t) \leftrightarrow \frac{1}{s}$$

$$e^{2t}u_s(t) \leftrightarrow \frac{1}{s-2}$$

$$(-t)e^{2t}u_s(t) \leftrightarrow \frac{d}{ds}[\frac{1}{s-2}] = \frac{0-1}{(s-2)^2} = \frac{-1}{(s-2)^2}$$

hence, $te^{2t}u_s(t) \leftrightarrow \dfrac{1}{(s-2)^2}$

Another solution)

$$t\,u_s(t) \leftrightarrow \frac{1}{s^2} \qquad e^{2t}\,t\,u_s(t) \leftrightarrow \frac{1}{(s-2)^2}$$

Example) $f(t) = te^{-t}\sin(2t)$

$$\sin(2t) \leftrightarrow \frac{2}{s^2+4}$$

$$e^{-t}\sin(2t) \leftrightarrow \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

$$te^{-t}\sin(2t) \leftrightarrow -\frac{d}{ds}\left[\frac{2}{s^2+2s+5}\right] = -\frac{0-2(2s+2)}{(s^2+2s+5)^2} = \frac{4(s+1)}{(s^2+2s+5)^2}$$

Another solution)

$$\sin(2t) \leftrightarrow \frac{2}{s^2+4}$$

$$t\sin(2t) \leftrightarrow -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] = -\frac{0-2(2s)}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$$

$$e^{-t}\,t\sin(2t) \leftrightarrow = \frac{4(s+1)}{((s+1)^2+4)^2} = \frac{4(s+1)}{(s^2+2s+5)^2}$$

Example) $f(t) = \sin^2(2t)$

$f'(t) = 2\sin(2t)\cos(2t)\cdot 2 = 2\sin(4t)$

where $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

$\therefore\ \sin(2A) = 2\sin(A)\cos(A)$

$L\{f'(t)\} = sF(s) - f(0)$ where $f(0) = \sin^2(0) = 0$

$$\therefore\ F(s) = \frac{1}{s}[L\{f'(t)\} + f(0)]$$

$$= \frac{1}{s}\left[\frac{2\cdot 4}{s^2+16} + 0\right] = \frac{8}{s(s^2+16)}$$

<Inverse Laplace TF> : Partial-Fraction Expansion)

$$f(t) \triangleq \frac{1}{2\pi j}\int_{\gamma - j\infty}^{\gamma + j\infty} F(s)e^{st}ds$$

※ Table 2.3 Laplace TF pairs

$$u(t) \leftrightarrow \frac{1}{s}$$

$$e^{-at} \leftrightarrow \frac{1}{s+a}$$

$$\sin(wt) \leftrightarrow \frac{w}{s^2+w^2}, \qquad \cos(wt) \leftrightarrow \frac{s}{s^2+w^2}$$

$$t^n \leftrightarrow \frac{n!}{s^{n+1}}$$

$$\frac{d^n f(t)}{dt^n} \leftrightarrow s^n F(s) - s^{n-1}f(0) - s^{n-2}f\,'(0) - \cdots - f^{n-1}(0)$$

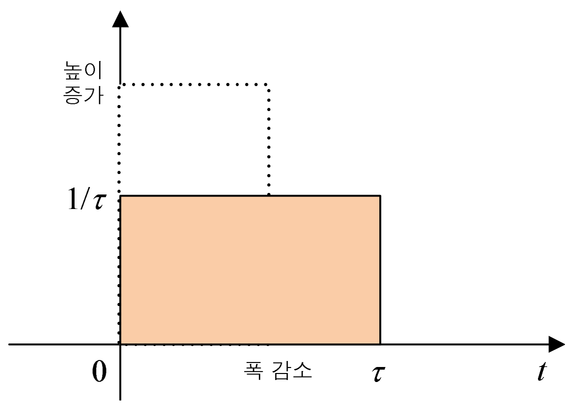
$$\int_{-\infty}^{t} f(t)\,dt \leftrightarrow \frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0} f(t)\,dt$$

$$e^{-at}\sin(wt) \leftrightarrow \frac{w}{(s+a)^2+w^2}$$

$$e^{-at}\cos(wt) \leftrightarrow \frac{s+a}{(s+a)^2+w^2}$$

$$\delta(t) \leftrightarrow 1$$

$$\delta(t) = \lim_{\tau \to 0}\frac{u(t)-u(t-\tau)}{\tau} = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$



$$\int_0^{\infty} \delta(t)\, e^{-st}\, dt = \lim_{\tau \to 0}\frac{1}{\tau}\int_0^{\infty} [u(t)-u(t-\tau)]e^{-st}\, dt$$

$$= \lim_{\tau \to 0}\frac{1}{\tau}\left[\int_0^{\infty} u(t)e^{-st}\, dt - \int_0^{\infty} u(t-\tau)e^{-st}\, dt\right]$$

$$= \lim_{\tau \to 0}\frac{1}{\tau}\left[\int_0^{\infty} e^{-st}\, dt - \int_{\tau}^{\infty} e^{-st}\, dt\right]$$

$$= \lim_{\tau \to 0}\frac{1}{\tau}\left[\frac{1}{s} - \frac{e^{-\tau s}}{s}\right]$$

$$= \lim_{\tau \to 0}\frac{1-e^{-\tau s}}{\tau s} \quad : \frac{0}{0} \text{ using L'Hopital's theorem}$$

$$= \lim_{\tau \to 0}\frac{+se^{-\tau s}}{s} = 1$$

<Partial fraction expansion>

$$F(s) = \frac{Q(s)}{P(s)} \triangleq \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}, \qquad n > m$$

(1) All poles of $F(s)$ are real and simple.

$$F(s) = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_n)}, \quad \text{where } s_1 \neq s_2 \neq \cdots \neq s_n$$

$$= \frac{k_1}{s+s_1} + \frac{k_2}{s+s_2} + \cdots + \frac{k_n}{s+s_n}$$

where $k_i = \left[(s+s_i)F(s)\right]_{s=-s_i}$

Example) $F(s) = \dfrac{5s+3}{(s+1)(s+2)(s+3)}$ To find $f(t)$

$$F(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2} + \frac{k_3}{s+3} = \frac{-1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3}$$

where
$$k_1 = \left[(s+1)F(s)\right]_{s=-1} = \left[\frac{5s+3}{(s+2)(s+3)}\right]_{s=-1} = \frac{-2}{2} = -1$$
$$k_2 = \left[(s+2)F(s)\right]_{s=-2} = \left[\frac{5s+3}{(s+1)(s+3)}\right]_{s=-2} = \frac{-7}{-1} = 7$$
$$k_3 = \left[(s+3)F(s)\right]_{s=-3} = \left[\frac{5s+3}{(s+1)(s+2)}\right]_{s=-3} = \frac{-12}{2} = -6$$

hence, $f(t) = -e^{-t}u(t) + 7e^{-2t}u(t) - 6e^{-3t}u(t)$

(2) Some poles of $F(s)$ are of multiple order r.

$$F(s) = \frac{Q(s)}{(s+s_1)(s+s_2)\cdots(s+s_{n-r})(s+s_i)^r}$$

$$= \frac{k_1}{s+s_1} + \frac{k_2}{s+s_2} + \cdots + \frac{k_{n-r}}{s+s_{n-r}} \qquad : n-r \text{ simple roots}$$

$$+ \frac{A_1}{s+s_i} + \frac{A_2}{(s+s_i)^2} + \cdots + \frac{A_r}{(s+s_i)^r} \qquad : r \text{ double roots}$$

where $k_j = \left[(s+s_j)F(s)\right]_{s=-s_j}$, for $j = 1, 2, \cdots, n-r$

$$A_r = \left[(s+s_i)^r F(s)\right]_{s=-s_i} \qquad A_{r-1} = \frac{d}{ds}\left[(s+s_i)^r F(s)\right]_{s=-s_i}$$

$$A_{r-2} = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+s_i)^r F(s)\right]_{s=-s_i} \qquad A_{r-3} = \frac{1}{3!}\frac{d^3}{ds^3}\left[(s+s_i)^r F(s)\right]_{s=-s_i}$$

$$\vdots \qquad \vdots$$

$$A_1 = \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}\left[(s+s_i)^r F(s)\right]_{s=-s_i}$$

Example) $F(s) = \dfrac{1}{s(s+1)^3(s+2)}$ Find $f(t)$

$$F(s) = \frac{k_0}{s} + \frac{k_2}{s+2} \qquad : \text{ simple roots}$$

$$+ \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3} \qquad : \text{ double roots}$$

$$= \frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1} - \frac{1}{(s+1)^3}$$

where

$$k_0 = \left[\frac{1}{(s+1)^3(s+2)}\right]_{s=0} = \frac{1}{2} \qquad k_2 = \left[\frac{1}{s(s+1)^3}\right]_{s=-2} = \frac{1}{2}$$

$$A_3 = \left[(s+1)^3F(s)\right]_{s=-1} = \left[\frac{1}{s(s+2)}\right]_{s=-1} = -1$$

$$A_2 = \left[\frac{d}{ds}\frac{1}{s(s+2)}\right]_{s=-1} = \left[\frac{-2(s+1)}{s^2(s+2)^2}\right]_{s=-1} = 0$$

$$A_1 = \frac{1}{2!}\left[\frac{d^2}{ds^2}\left(\frac{1}{s(s+2)}\right)\right]_{s=-1} = \frac{1}{2!}\left[\frac{d}{ds}\left(\frac{-2(s+1)}{s^2(s+2)^2}\right)\right]_{s=-1}$$

$$= \frac{-2}{2!}\left[\frac{s^2(s+2)^2-(s+1)[2s(s+2)^2+s^2\cdot 2(s+2)]}{s^4(s+2)^4}\right]_{s=-1}$$

$$= -1$$

hence, $f(t) = \dfrac{1}{2}u(t) + \dfrac{1}{2}e^{-2t}u(t) - e^{-t}u(t) - \dfrac{t^2}{2}e^{-t}u(t)$

(3) Simple complex conjugate poles.

Example) $F(s) = \dfrac{2s}{(s+1)(s^2+2s+5)} = \dfrac{2s}{(s+1)(s+1+j2)(s+1-j2)}$

$$F(s) = \frac{k_1}{s+1} + \frac{k_{1+j2}}{s+1+j2} + \frac{k_{1-j2}}{s+1-j2} = \frac{-1/2}{s+1} + \frac{(1+2j)/4}{s+1+2j} + \frac{(1-2j)/4}{s+1-2j}$$

where

$$k_1 = \left[\frac{2s}{s^2+2s+5}\right]_{s=-1} = -\frac{1}{2}$$

$$k_{1+2j} = \left[\frac{2s}{(s+1)(s+1-2j)}\right]_{s=-1-2j} = \frac{1+2j}{4}$$

$$k_{1-2j} = \left[\frac{2s}{(s+1)(s+1+2j)}\right]_{s=-1+2j} = \frac{1-2j}{4}$$

Hence,

$$
\begin{aligned}
f(t) &= -\frac{1}{2}e^{-t}u(t) + \frac{1+2j}{4}e^{-(1+2j)t}u(t) + \frac{1-2j}{4}e^{-(1-2j)t}u(t) \\
&= -\frac{1}{2}e^{-t}u(t) + \frac{1}{4}e^{-t}\left[e^{-j2t} + e^{+j2t}\right]u(t) + \frac{2j}{4}e^{-t}\left[e^{-j2t} - e^{+j2t}\right]u(t) \\
&= -\frac{1}{2}e^{-t}u(t) + \frac{1\cdot 2}{4}e^{-t}\left[\frac{e^{+j2t}+e^{-j2t}}{2}\right]u(t) - \frac{2j\cdot 2j}{4}e^{-t}\left[\frac{e^{+j2t}-e^{-j2t}}{2j}\right]u(t) \\
&= -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-t}\cos(2t) + e^{-t}\sin(2t)
\end{aligned}
$$

(Another solution)

$$
F(s) = \frac{2s}{(s+1)(s^2+2s+5)} = \frac{k_1}{s+1} + \frac{as+b}{s^2+2s+5} = \frac{1/2}{s+1} + \frac{1}{2}\frac{s+5}{s^2+2s+5}
$$

Compare the coefficients

$$
\begin{aligned}
2s &= k_1(s^2+2s+5) + (s+1)(as+b) \\
&= (k_1+a)s^2 + (a+b+2k_1)s + 5k_1 + b
\end{aligned}
$$

where $k_1 = [(s+1)F(s)]_{s=-1} = -\frac{1}{2}$

$$
\begin{aligned}
k_1 + a = -\frac{1}{2} + a = 0 \quad &\Rightarrow \quad a = \frac{1}{2} \\
a + b + 2k_1 = a + b - 1 = 2 \quad &\Rightarrow \quad b = \frac{5}{2} \\
\text{or} \quad 5k_1 + b = -\frac{5}{2} + b = 0 \quad &\Rightarrow \quad b = \frac{5}{2}
\end{aligned}
$$

hence,

$$
\begin{aligned}
f(t) &= -\frac{1}{2}e^{-t} + \frac{1}{2}\left[e^{-t}\cos(2t) + 2e^{-t}\sin(2t)\right] \\
&= \frac{1}{2}e^{-t}\left[-1 + \cos(2t) + 2\sin(2t)\right]
\end{aligned}
$$

where

$$
\frac{s+5}{s^2+2s+5} = \frac{s+5}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{2\cdot 2}{(s+1)^2+2^2}
$$

$$
\leftrightarrow \quad e^{-t}\cos(2t) + 2e^{-t}\sin(2t)
$$

<Mass-Spring-Damper system>

$$
M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + Ky = r(t), \qquad \text{find } y(t).
$$

where $r(t) = 0,\ y(0) = y_0\ ,\quad y'(0) = \left.\frac{dy}{dt}\right|_{t=0} = 0,\qquad M,\ b,\ K$: constant

solution)

taking Laplace TF

$$M[s^2 Y(s) - s\,y(0) - y'(0)] + b[s\,Y(s) - y(0)] + KY(s) = R(s)$$

$$[Ms^2 + bs + K]\,Y(s) = (Ms + b)\,y_0$$

$$\therefore\ Y(s) = \frac{Ms+b}{Ms^2+bs+K}\,y_0$$: natural response $[r(t)=0]$

characteristic eq. $Ms^2 + bs + K = 0$

<General 2nd order system>

Natural resp. (transient resp.) if $r(t)=0$.

$$Y(s) = \frac{s+2\zeta w_n}{s^2+2\zeta w_n s + w_n^2}\,y_0 \equiv \frac{s+\frac{b}{M}}{s^2+\frac{b}{M}s+\frac{K}{M}}\,y_0$$

where natural freq. : $w_n = \sqrt{K/M}$

damping ratio. : $\zeta = b/(2\sqrt{MK})$

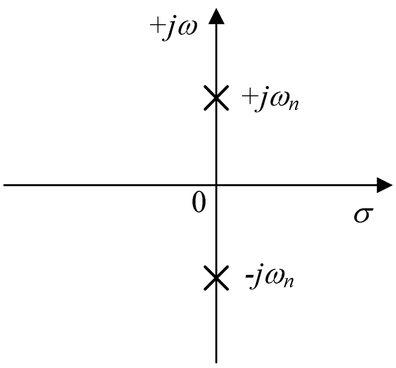
char. eg $s^2 + 2\zeta w_n s + w_n^2 = 0$

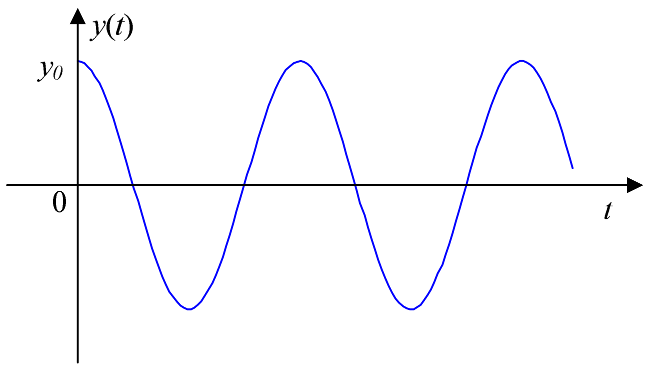
poles $s_{1,2} = -\zeta w_n \pm w_n\sqrt{\zeta^2-1}$

(1) No damping $(\zeta = 0)$

char. eg. $s^2 + w_n^2 = 0 \quad \Rightarrow \quad s_{1,2} = \pm j w_n$

$$Y(s) = \frac{s}{s^2+w_n^2}\,y_0 \quad \Rightarrow \quad y(t) = y_0 \cos(w_n t)$$



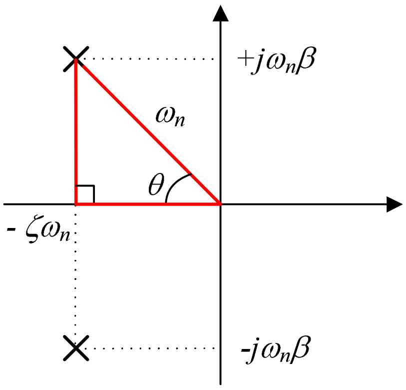


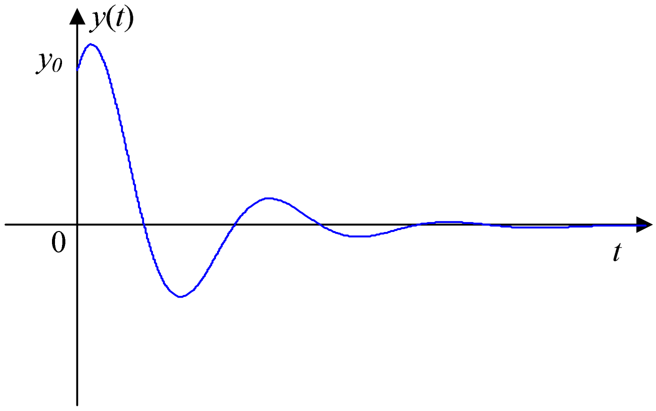
Example) $K/M = 2. \quad b/M = 0. \quad \therefore \zeta = 0$

$$Y(s) = \frac{s}{s^2+2}\,y_0 \qquad \therefore\ y(t) = y_0 \cos\ \sqrt{2}\,t$$

(2) Underdamping $(0 < \zeta < 1)$

char. eg. $s^2 + 2\zeta w_n s + w_n^2 = 0 \quad \Rightarrow s_{1,2} = -\zeta w_n \pm j w_n \beta$ where $\beta = \sqrt{1-\zeta^2}$





$$\cos(\theta) = \frac{\zeta w_n}{w_n} = \zeta \qquad \sin(\theta) = \frac{\beta w_n}{w_n} = \beta \qquad \tan(\theta) = \frac{w_n \beta}{\zeta w_n} = \frac{\beta}{\zeta}$$

$$\begin{aligned} s^2 + 2\zeta w_n s + w_n^2 &= s^2 + 2\zeta w_n s + \zeta^2 w_n^2 - \zeta^2 w_n^2 + w_n^2, \qquad \text{where } \beta = \sqrt{1-\zeta^2} \\ &= (s+\zeta w_n)^2 + w_n^2(1-\zeta^2) \\ &= (s+\zeta w_n)^2 + (w_n\beta)^2 \end{aligned}$$

$$\begin{aligned} Y(s) = \frac{s + 2\zeta w_n}{s^2 + 2\zeta w_n s + w_n^2} y_0 &= \frac{(s+\zeta w_n) + (\zeta w_n)}{(s+\zeta w_n)^2 + (w_n\beta)^2} y_0 \\ &= \frac{s+\zeta w_n}{(s+\zeta w_n)^2 + (w_n\beta)^2} y_0 + \frac{\zeta}{\beta} \cdot \frac{w_n\beta}{(s+\zeta w_n)^2 + (w_n\beta)^2} y_0 \end{aligned}$$

Taking Inverse Laplace TF

$$\begin{aligned} y(t) &= y_0 e^{-\zeta w_n t} \cos(w_n\beta t) + y_0 \frac{\zeta}{\beta} e^{-\zeta w_n t} \sin(w_n\beta t) \\ &= y_0 e^{-\zeta w_n t} \frac{1}{\beta} [\beta \cos(w_n\beta t) + \zeta \sin(w_n\beta t)] \qquad \Leftarrow \beta = \sin(\theta), \quad \zeta = \cos(\theta) \\ &= \frac{y_0}{\beta} e^{-\zeta w_n t} \sin(w_n\beta t + \theta) \end{aligned}$$

Example) $\frac{K}{M} = 2$ and $\frac{b}{M} = 2$

$$Y(s) = \frac{s+2}{s^2+2s+2} y_0 = \frac{s+1+1}{(s+1)^2+1} y_0$$

$$y(t) = y_0 e^{-t} [\cos(t) + \sin(t)] = y_0 \sqrt{2} e^{-t} \sin(t + 45^\circ)$$

where

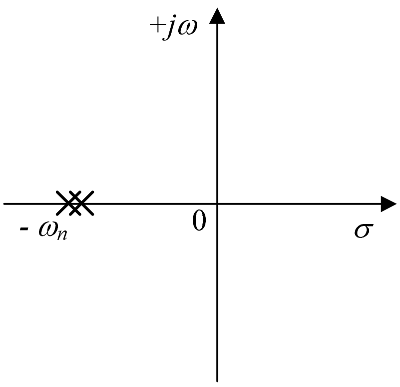
$$\begin{aligned} \cos(t) + \sin(t) &= \sqrt{2} \left[\frac{1}{\sqrt{2}} \cos(t) + \frac{1}{\sqrt{2}} \sin(t) \right] \\ &= \sqrt{2} \sin(t + 45^\circ) \end{aligned}$$

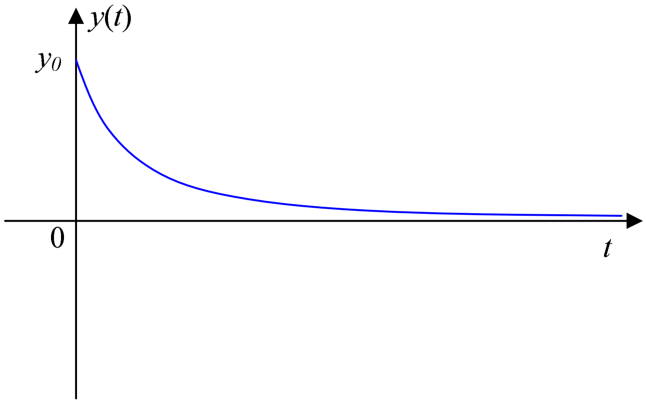
(3) Critical damping ($\zeta = 1$)

char. eg. $s^2 + 2w_n s + w_n^2 = (s + w_n)^2 = 0 \quad \Rightarrow s_{1,2} = -w_n$: double root

$$Y(s) = \frac{s + 2w_n}{s^2 + 2w_n s + w_n^2} y_0 = \frac{s + w_n + w_n}{(s + w_n)^2} y_0 = \left[\frac{1}{s + w_n} + \frac{w_n}{(s + w_n)^2} \right] y_0$$

$$\Rightarrow y(t) = y_0 \left(e^{-w_n t} + e^{-w_n t} w_n t \right) = y_0 e^{-w_n t} (1 + w_n t)$$





Example) $\frac{K}{M} = 2$ and $\frac{f}{M} = 2\sqrt{2}$

$$Y(s) = \frac{s + 2\sqrt{2}}{s^2 + 2\sqrt{2}s + 2} y_0 = \frac{s + 2\sqrt{2}}{(s + \sqrt{2})^2} y_0 = \left[\frac{1}{(s + \sqrt{2})} + \frac{\sqrt{2}}{(s + \sqrt{2})^2} \right] y_0$$

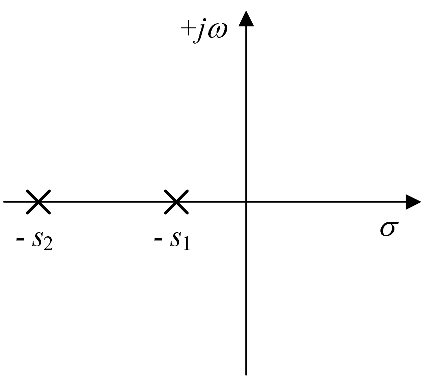
$$\Rightarrow y(t) = y_0 e^{-\sqrt{2}t} (1 + \sqrt{2}t)$$

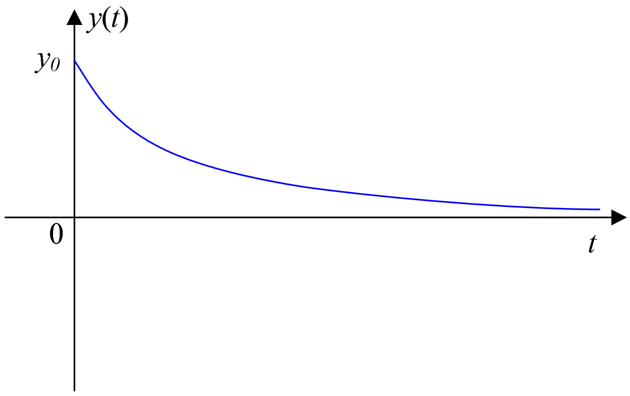
(4) Overdamping ($\zeta > 1$)

char. eg. $s^2 + 2\zeta w_n s + w_n^2 = 0 \quad s_{1,2} = -\zeta w_n \pm w_n \sqrt{\zeta^2 - 1}$: real simple pole.

$$Y(s) = \frac{s + 2w_n}{s^2 + 2\zeta w_n s + w_n^2} y_0 = \left[\frac{k_1}{s + s_1} + \frac{k_2}{s + s_2} \right] y_0$$

$$\Rightarrow y(t) = y_0 \left(k_1 e^{-s_1 t} + k_2 e^{-s_2 t} \right)$$



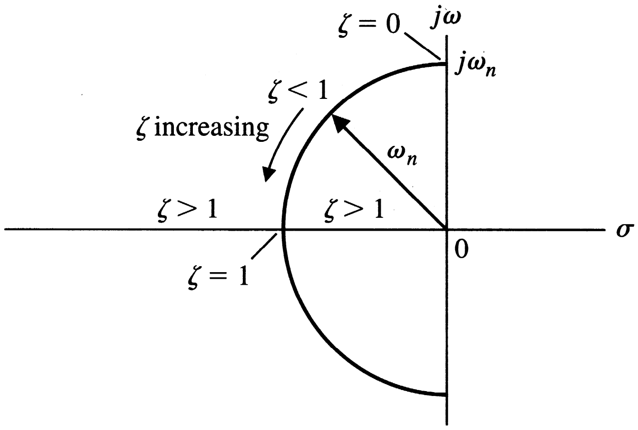


Example) $\dfrac{K}{M} = 2$ and $\dfrac{f}{M} = 3$.

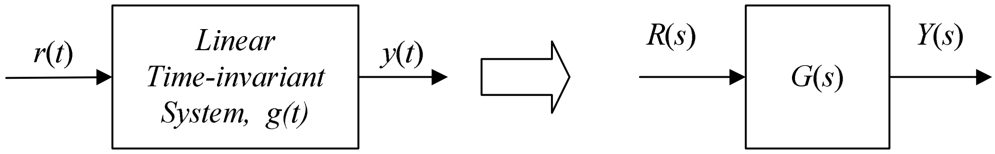
$$Y(s) = \frac{s+3}{s^2+3s+2}y_0 = \left(\frac{2}{s+1} - \frac{1}{s+2}\right)y_0$$

$$y(t) = y_0\left(2e^{-t} - e^{-2t}\right)$$

※ Root locus as ζ varies with w_n constant



2.5 The Transfer Function of Liner Systems



$y(t) = g(t)*r(t)$ $\qquad\qquad$ $Y(s) = G(s)\,R(s)$

$$G(s) \equiv \frac{Y(s)}{R(s)}$$: Transfer fn with all initial conditions = 0

<Mass-Spring-Damper system>

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + Ky = r(t)$$

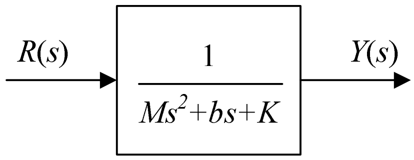
taking Laplace TF

$$M[s^2 Y(s) - s\,y(0) - y'(0)] + b[s\,Y(s) - y(0)] + KY(s) = R(s)$$

Transfer fn. with all initial conditions zero

$$[Ms^2 + bs + K]\,Y(s) = R(s)$$

$$G(s) \equiv \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + K}$$

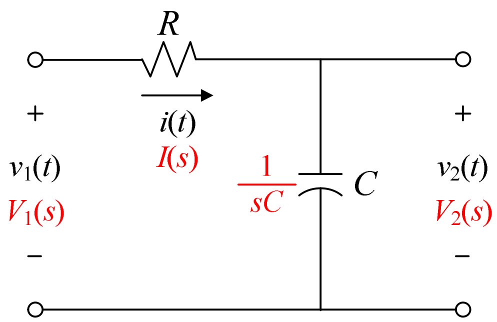


characteristic eq. $\quad Ms^2 + bs + K = 0$

pole $\quad s_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4MK}}{2M}$

zero : none

<RC circuit>



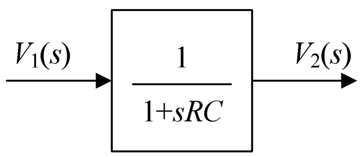
$$V_2(s) = \frac{1}{sC}\,I(s)$$

$$\text{TF} \quad G_1(s) = \frac{V_2(s)}{I(s)} = \frac{1}{sC}$$

$$V_2(s) = \frac{1/sC}{R + 1/sC}\,V_1(s)$$

$$\text{TF} \quad G_2(s) = \frac{V_2(s)}{V_1(s)} = \frac{1}{1 + sRC}$$

where $\tau = RC$: time constant



Example 2.2) Solution of differential eq.

$$\frac{d^2y}{dt}+4\frac{dy}{dt}+3y = 2r(t),$$

Initial conditions $y(0)=1,\quad y'(0)=0.\quad r(t)=1,\quad t\geq 0$

sol)

$$s^2Y(s)-sy(0)-y'(0)+4[sY(s)-y(0)]+3Y(s) = 2R(s)$$

$$(s^2+4s+3)Y(s)-(s+4)y(0) = 2R(s)$$

$$Y(s)=\frac{s+4}{(s+1)(s+3)}y(0)+\frac{2}{(s+1)(s+3)}R(s) \quad \text{where } y(0)=1 \text{ and } R(s)=\frac{1}{s}$$

$$=\frac{s+4}{(s+1)(s+3)}+\frac{2}{s(s+1)(s+3)}$$

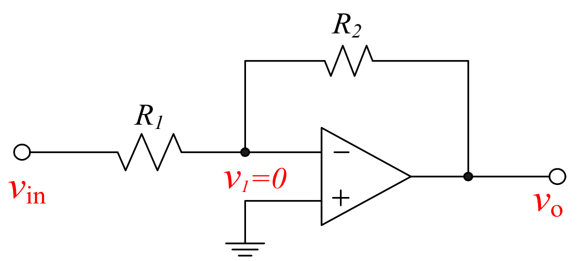
$$=\frac{s^2+4s+2}{s(s+1)(s+3)}$$

$$=\frac{k_0}{s}+\frac{k_1}{s+1}+\frac{k_3}{s+3}$$

where $k_0=\left[\frac{s^2+4s+2}{(s+1)(s+3)}\right]_{s=0}=\frac{2}{3},\qquad k_1=\frac{1}{2},\qquad k_3=-\frac{1}{6}$

Hence, $y(t)=\left[\frac{2}{3}+\frac{1}{2}e^{-t}-\frac{1}{6}e^{-3t}\right]u(t)$

Example) Inverting amplifier with op amp.

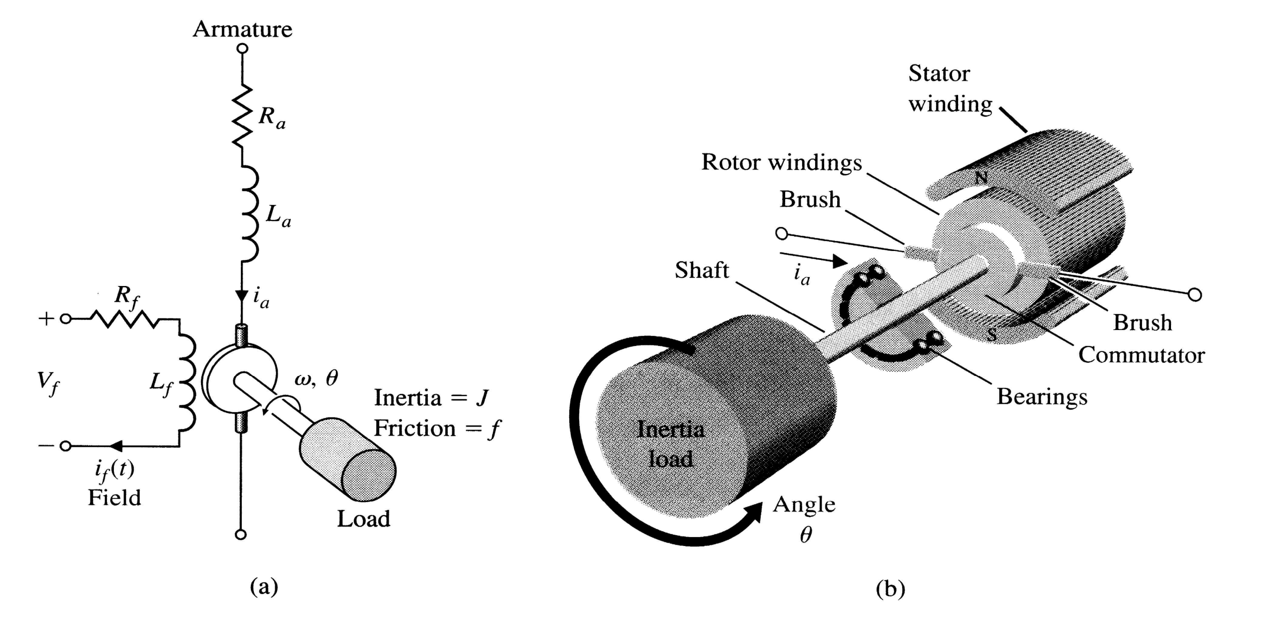


$$\frac{v_1-v_{in}}{R_1}+\frac{v_1-v_o}{R_2}=0 \qquad \text{where } v_1=v_2=0$$

$$-\frac{v_{in}}{R_1}-\frac{v_o}{R_2}=0$$

Hence, $\dfrac{v_0}{v_{in}}=-\dfrac{R_2}{R_1}$

Example 2.5) Transfer Function of dc motor



※ motor torque

$$T_m(t) = K_1 \phi i_a(t) \quad \text{where } \phi = K_f i_f(t)$$
$$= K_1 K_f i_f(t) i_a(t)$$

(1) field current-controlled dc motor

$[i_a(t) = \text{constant}, \; i_f(t) = \text{input}]$

Motor Torque

$$T_m(t) = K_m i_f(t) = T_L(t) + T_d(t)$$

where $T_L(t) = J\frac{d^2\theta(t)}{dt^2} + f\frac{d\theta(t)}{dt}$: Load Torque,

$T_D(t) = 0$: Disturbance Torque

Field Voltage

$$v_f(t) = R_f i_f(t) + L_f \frac{d i_f(t)}{dt}$$

Taking Laplace TF

$$T_m(s) = K_m I_f(s) = T_L(s)$$

$$V_f(s) = R_f I_f(s) + L_f s I_f(s) \qquad \therefore \; I_f(s) = \frac{1}{L_f s + R_f} V_f(s)$$

$$T_L(s) = Js^2\theta(s) + f s\theta(s)$$

$$\Rightarrow \quad \theta(s) = \frac{T_L(s)}{s(Js+f)} = \frac{K_m I_f(s)}{s(Js+f)} = \frac{K_m}{s(Js+f)(L_f s + R_f)} V_f(s)$$

Transfer Function

$$G(s) \equiv \frac{\theta(s)}{V_f(s)} = \frac{K_m}{s\,(Js+f)(L_f\,s+R_f)}$$

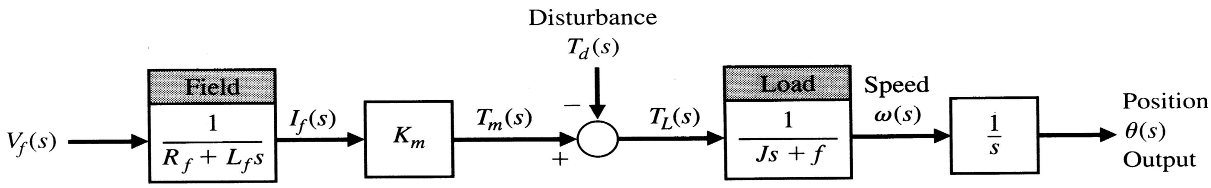
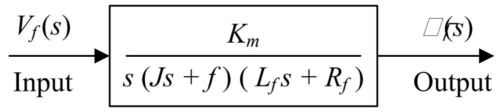


Fig. 2.19 Block diagram model of field controlled dc motor



(2) armature current controlled dc motor

[$i_f(t)$: constant, $i_a(t)$: input]

Motor Torque

$$T_m(t) = K_m i_a(t)$$

$$V_a(t) = R_a i_a(t) + L_a \frac{di_a(t)}{dt} + V_b(t)$$

Where $v_b(t) = K_b\, w(t) \equiv K_b \frac{d\theta(t)}{dt}$: back electromotive-force voltage

$$T_L(t) = J\frac{d^2\theta(t)}{dt^2} + f\frac{d\theta(t)}{dt} \triangleq T_m(t) - T_d(t)$$

Taking Laplace TF at both side

$$T_m(s) = K_m I_a(s)$$

$$V_a(s) = R_a I_a(s) + L_a\, s\, I_a(s) + K_b W(s) \quad \text{where } W(s) = s\,\theta(s)$$

$$\therefore)\ I_a(s) = \frac{1}{L_a\, s + R_a} V_a(s) - \frac{K_b\, s}{L_a\, s + R_a}\theta(s)$$

$$T_L(s) = Js^2\theta(s) + f\, s\,\theta(s)$$

$$\therefore)\quad \theta(s) = \frac{T_L(s)}{s(Js+f)} = \frac{K_m\, I_a(s)}{s\,(Js+f)} \quad \text{where } T_L(s) = K_m\, I_a(s) \text{ if } T_D(s) = 0$$

$$= \frac{K_m\, V_a(s)}{s\,(Js+f)(L_a\, s + R_a)} - \frac{K_m K_b\, s\,\theta(s)}{s\,(Js+f)(L_a\, s + R_a)}$$

If arranged

$$\frac{s(Js+f)(L_a s+R_a)+K_m K_b s}{s(Js+f)(L_a s+R_a)}\theta(s) = \frac{K_m}{s(Js+f)(L_a s+R_a)} V_a(s)$$

Transfer Function

$$G(s)=\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(Js+f)(L_a s+R_a)+K_m K_b]}$$
$$\equiv \frac{K}{s(s^2+2\zeta w_n s+w_n^2)}$$

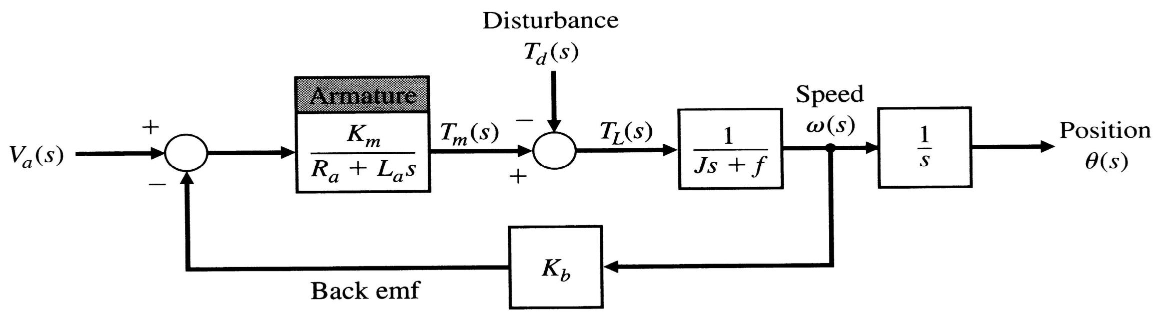
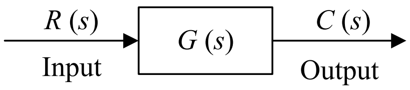


Fig 2.20 Armature-controlled dc motor

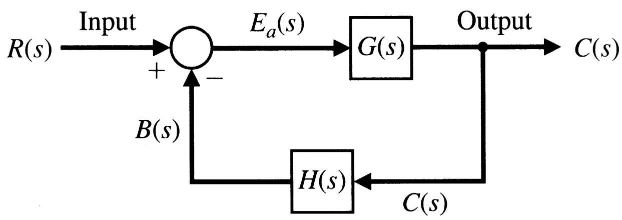
2.6 Block diagram models

: unidirectional operational blocks.



$C(s) = G(s) \cdot R(s) \qquad G(s) = \frac{C(s)}{R(s)}$: Transfer fn

※ Negative f/b control system : Fig 2.25.



Error signal : $E_a(s) = R(s) - HC(s)$

output

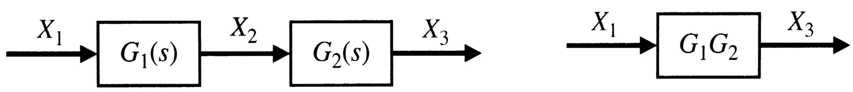
$$C(s) = G(s) \cdot E_a(s) = G(s)[R(s) - HC(s)] = GR(s) - GHC(s)$$

$$[1 + GH(s)]C(s) = G(s)R(s)$$

$$C(s) = \frac{G(s)}{1 + GH(s)} R(s)$$

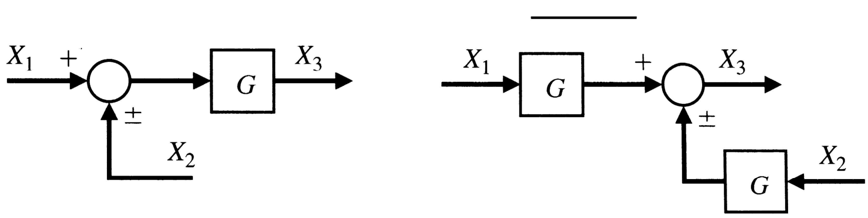
Table 2.6 Block diagram transformations..

(1) cascade connection



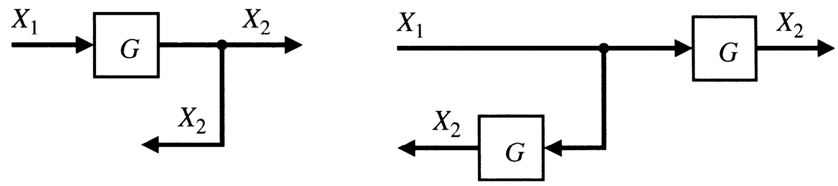
$X_2 = G_1 X_1 \qquad X_3 = G_2 X_2 \qquad \therefore)\ X_3 = G_1 G_2 X_1$

(2) Moving a summing pt behind a block

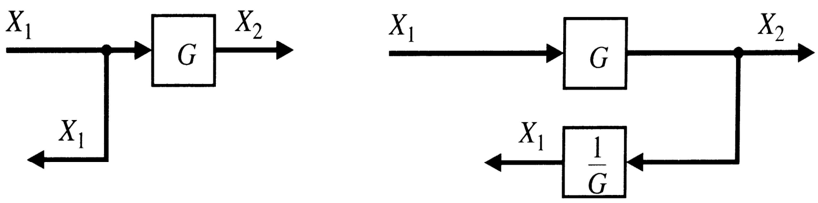


$X_3 = G(X_1 \pm X_2) \quad \Rightarrow \quad X_3 = GX_1 \pm GX_2$

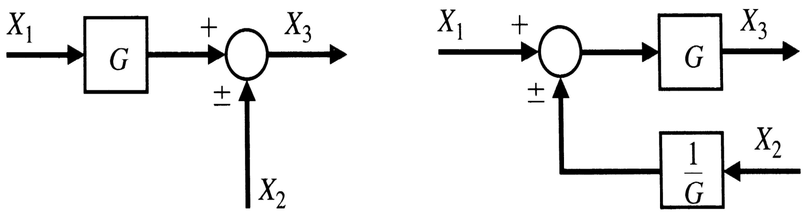
(3) Moving a pick-off pt ahead of a block



(4) Moving a pick-off behind a block



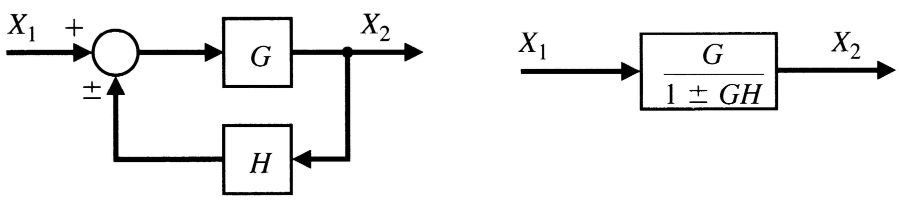
(5) Moving a summing pt ahead of a block



$X_3 = GX_1 \pm X_2$ $\qquad\qquad$ $X_3 \ = \ G(X_1 \pm \frac{1}{G} X_2)$

$\quad = G(X_1 + \frac{1}{G} X_2)$

(6) Eliminating a f/b loop.



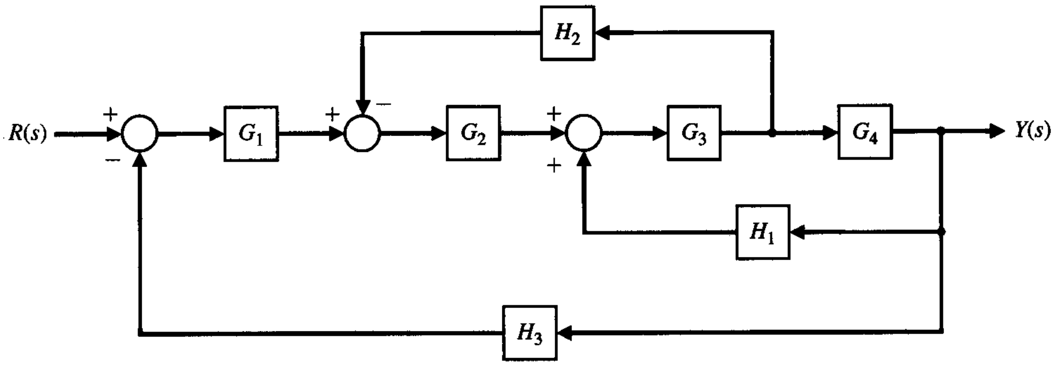
$E(s) \ = \ X_1 \ \pm \ HX_2$

$X_2 \ = \ GE(s) \ = \ GX_1(s) \ \pm \ GHX_2(s)$

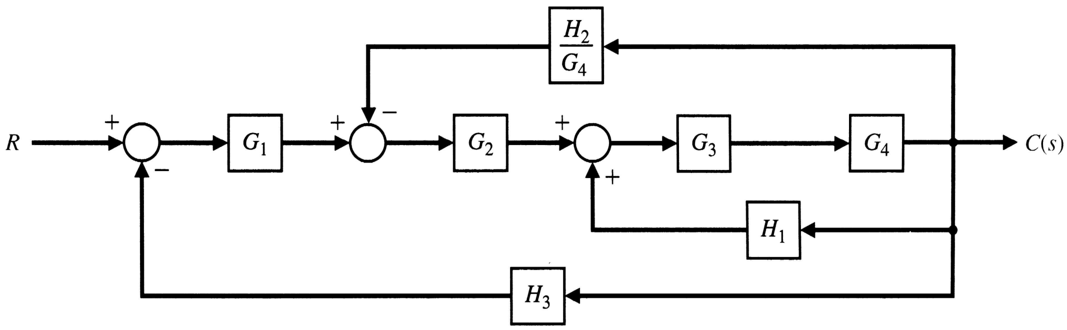
$(1 \mp GH\ )X_2(s) \ = \ GX_1(s)$

$$X_2(s) = \frac{G}{1 \mp GH} X_1(s)$$

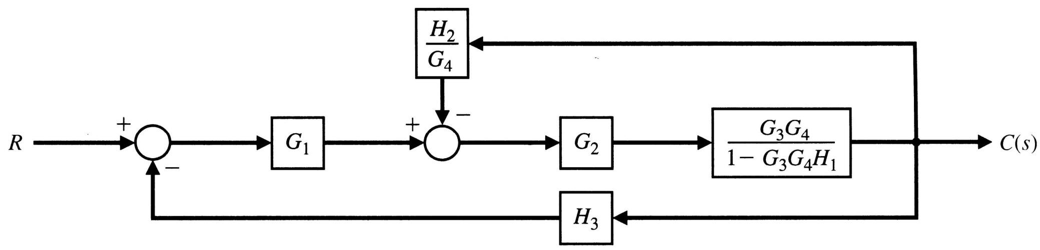
Example 2.7) Block diagram reduction : Fig 2.26



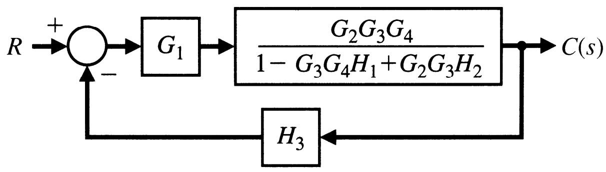
(1) Moving a pick off point ($G_3 - G_4$) behind G_4 using rule 4.



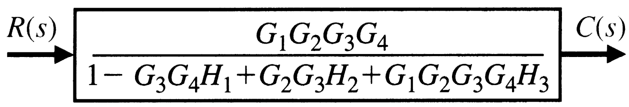
(2) Eliminating $G_3 \cdot G_4 \cdot H_1$ loop using rule 6.



(3) Eliminating $G_2 \cdot \dfrac{G_3 G_4}{1 - G_3 G_4 H_1} \cdot \dfrac{H_2}{G_4}$ loop using rule 6.

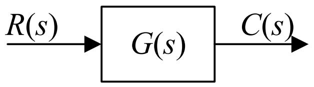


(4) Eliminating final loop.

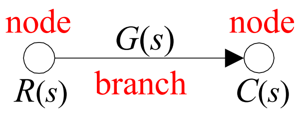


2.7 Signal Flow Graph Models..

$C(s) = G(s) \cdot R(s)$: linear algebraic eg.



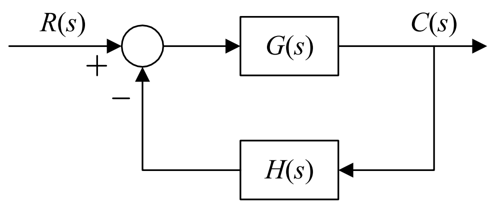
: Block diagram



: Signal Flow Graph

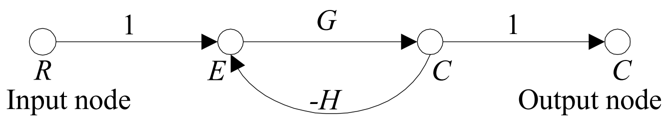
where R(s) and C(s) : System signal
$G(s)$: Gain

Example)



$E(s) = R(s) - H(s)\, C(s)$

$C(s) = G(s)\, E(s)$



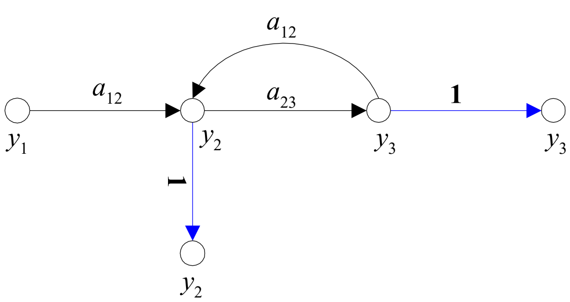
<notations>

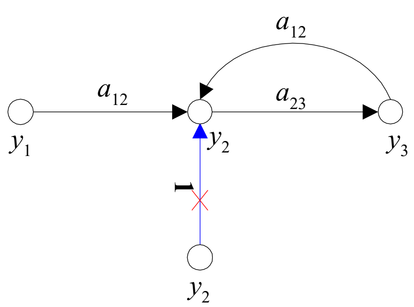
(1) input node(source) : 밖으로 나가는 가지들만을 갖는 마디.

(2) output node(sink) : 들어오는 가지들만을 갖는 마디.

Example) $y_2 = a_{12}y_1 + a_{32}y_3$

$y_3 = a_{23}y_2$





Input node : y_1

Output node : y_2, y_3 (any node)

※ 비입력마디를 입력마디로 만들 수 없음.

$y_2 \neq a_{12}\, y_1 + a_{32}\, y_3 \quad + y_2$: Not

(3) path(경로) : 같은 방향의 연속적인 가지들의 집합

(4) forward path : input node → output node (2번 이상 거쳐서는 안됨)

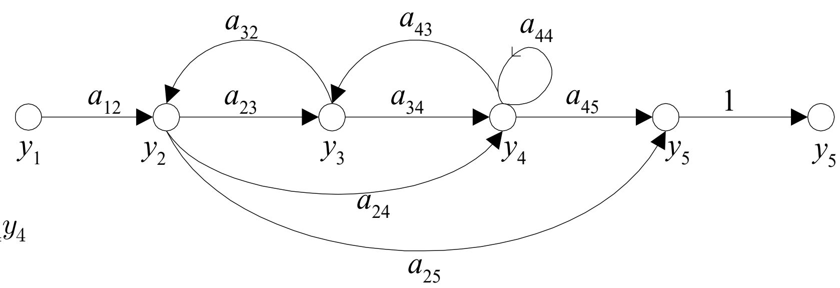
(5) path gain(경로 이득)

(6) loop : 어떤 마디에서 시작하여 그 마디에서 끝나는 경로

(7) loop gain

Example)

$$y_2 = a_{12}y_1 + a_{32}y_3$$
$$y_3 = a_{23}y_2 + a_{43}y_4$$
$$y_4 = a_{24}y_2 + a_{34}y_3 + a_{44}y_4$$
$$y_5 = a_{25}y_2 + a_{45}y_4$$



(1) input node : y_1

(2) output node : y_5 (any node)

(3) forward path

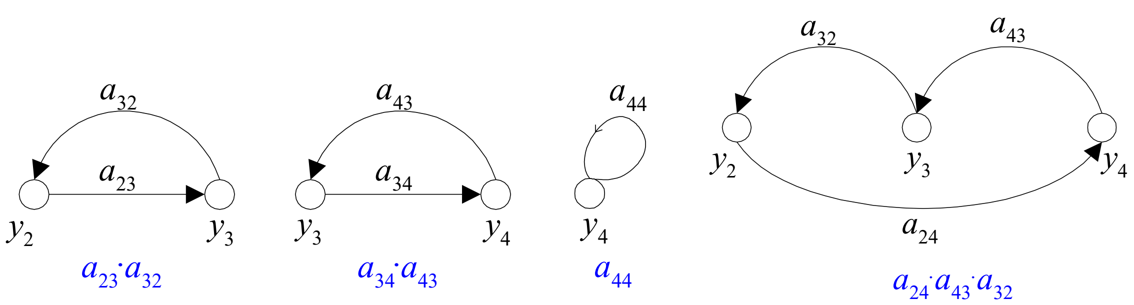
$y_1 \rightarrow y_2$: $y_1 - y_2$: a_{12}

$$y_1 \rightarrow y_3 : \begin{cases} y_1 - y_2 - y_3 & : a_{12} \cdot a_{23} \\ y_1 - y_2 - y_4 - y_3 & : a_{12} \cdot a_{24} \cdot a_{43} \end{cases}$$

$$y_1 \rightarrow y_4 : \begin{cases} y_1 - y_2 - y_3 - y_4 & : a_{12} \cdot a_{23} \cdot a_{34} \\ y_1 - y_2 - y_4 & : a_{12} \cdot a_{24} \end{cases}$$

$$y_1 \rightarrow y_5 : \begin{cases} y_1 - y_2 - y_3 - y_4 - y_5 & : a_{12} \cdot a_{23} \cdot a_{34} \cdot a_{45} \\ y_1 - y_2 - y_4 - y_5 & : a_{12} \cdot a_{24} \cdot a_{45} \\ y_1 - y_2 - y_5 & : a_{12} \cdot a_{25} \end{cases}$$

(4)loop



※ Mason's loop rule (이득 공식)

$$T \triangleq \frac{y_{out}}{y_{in}} = \frac{\sum_{k=1}^{N} P_k \triangle k}{\triangle}$$

where

T : gain

N : # of forward path btn $y_{\text{in}} \sim y_{\text{out}}$.

P_k : gain of k-th forward path.

$\triangle$ = 1 - (단일 loop 이득의 합)
+ (2개의 비접촉 loop들의 이득곱의 합)
- (3개의 비접촉 loop들의 이득곱의 합)
+ (4개의비접촉 loop들의 이득곱의 합)
- (5개의 비접촉의 loop들의 합)
+ .....

$\triangle_k$: cofactor of the path P_k .
(k번째 forward path를 제외한 신호흐름 선도에서의 △)

Example) previous example, $\frac{y_5}{y_1} = ?$

sol)

# of forward path $N = 3$

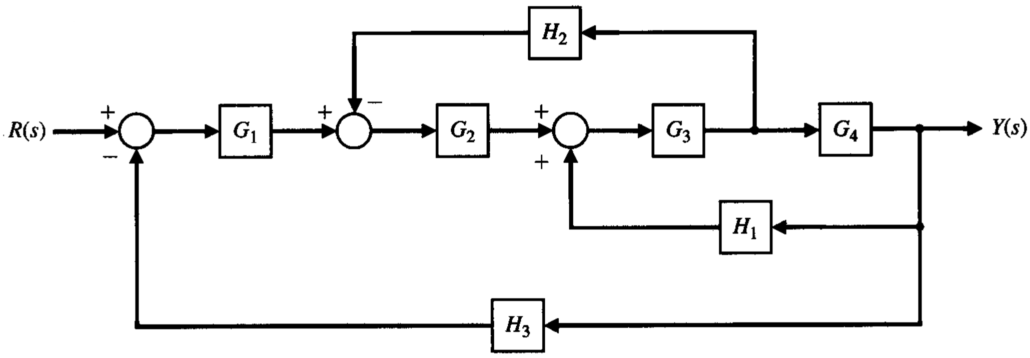
$$\begin{cases} y_1 - y_2 - y_3 - y_4 - y_5 : & P_1 = a_{12} \cdot a_{23} \cdot a_{34} \cdot a_{45} & : \triangle_1 = 1 \\ y_1 - y_2 - y_4 - y_5 & : P_2 = a_{12} \cdot a_{24} \cdot a_{45} & : \triangle_2 = 1 \\ y_1 - y_2 - y_5 & : P_3 = a_{12} \cdot a_{25} & : \triangle_3 = 1 - a_{34} \cdot a_{43} \cdot a_{44} \end{cases}$$

$$\triangle = 1 - (a_{23} \cdot a_{32} + a_{34} \cdot a_{43} + a_{44} + a_{24} \cdot a_{43} \cdot a_{32}) + (a_{23} \cdot a_{32} \cdot a_{44})$$

Hence, Gain is

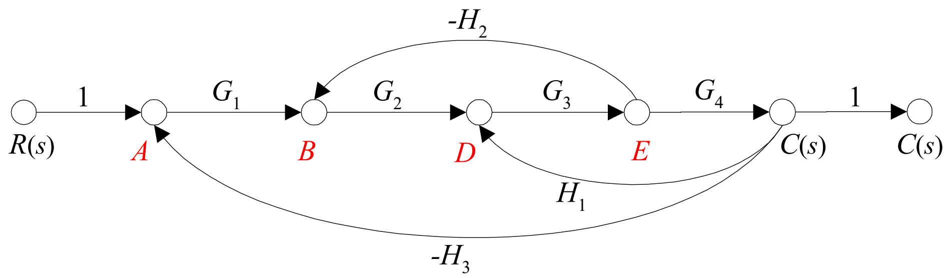
$$T = \frac{y_5}{y_1}$$
$$= \frac{1}{\triangle}(P_1 \triangle_1 + P_2 \triangle_2 + P_2 \triangle_3)$$
$$= \frac{1}{\triangle}(a_{12} \cdot a_{23} \cdot a_{34} \cdot a_{45} + a_{12} \cdot a_{24} \cdot a_{45} + a_{12} \cdot a_{25}$$
$$- a_{12} \cdot a_{25} \cdot a_{34} \cdot a_{43} - a_{12} \cdot a_{25} \cdot a_{44})$$

Example) TF of Multiple loop system ~ Fig 2.26



sol)

Signal flow graph



# of forward path : $N=1$

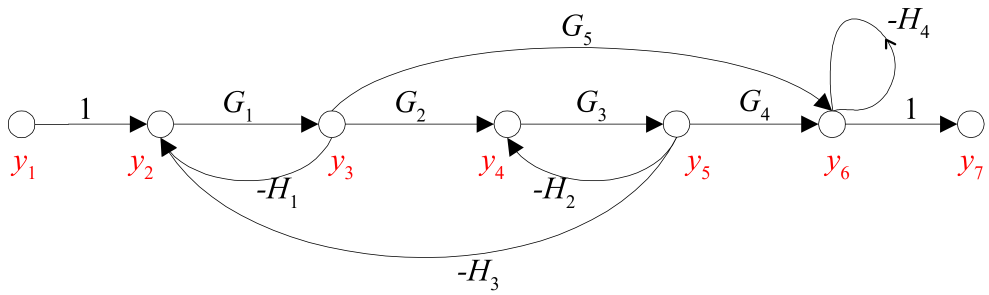
$P_1 = G_1 G_2 G_3 G_4$ and $\triangle_1 = 1$

$$\triangle = 1-[G_2 G_3(-H_2)+G_3 G_4 H_1 + G_1 G_2 G_3(-H_3)]$$
$$= 1+G_2 G_3 H_2 - G_3 G_4 H_1 + G_1 G_2 G_3 G_4 H_3$$

Hence

$$\frac{C(s)}{R(s)} = \frac{P_1 \triangle_1}{\triangle} = \frac{G_1 G_2 G_3 G_4}{\triangle}$$

Example) KUO



$$\triangle = 1-(-G_1H_1 - G_3H_2 - H_4 - G_1G_2G_3H_3) + (G_1H_1G_3H_2 + G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4) - (-G_1H_1G_3H_2H_4)$$

$$= 1 + G_1H_1 + G_3H_2 + H_4 + G_1G_2G_3H_3 + G_1H_1G_3H_2 + G_1H_1H_4 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1H_1G_3H_2H_4$$

(1) $\dfrac{y_2}{y_1} = \dfrac{P_1\triangle_1}{\triangle}$

where $P_1 = 1$

$$\triangle_1 = 1-(-G_3H_2 - H_4) + (G_3H_2H_4) = 1 + G_3H_2 + H_4 + G_3H_2H_4$$

Hence $\dfrac{y_2}{y_1} = \dfrac{1}{\triangle}(1 + G_3H_2 + H_4 + G_3H_2H_4)$

(2) $\dfrac{y_4}{y_1} = \dfrac{P_1\triangle_1}{\triangle} = \dfrac{1}{\triangle}[G_1G_2(1+H_4)]$

where $P_1 = G_1G_2$

$$\triangle_1 = 1-(-H_4) = 1 + H_4$$

(3) $\dfrac{y_6}{y_1} = \dfrac{M_1\triangle_1 + M_2\triangle_2}{\triangle} = \dfrac{1}{\triangle}[G_1G_2G_3G_4 + G_1G_5(1+G_3H_2)]$

where $P_1 = G_1G_2G_3G_4$ $P_2 = G_1G_5$

$\triangle_1 = 1$ $\triangle_2 = 1 + G_3H_2$

(4) $\dfrac{y_6}{y_2} = \dfrac{y_6/y_1}{y_2/y_1} = \dfrac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2)}{1 + G_3H_2 + H_4 + G_3H_2H_4}$